Shape preservation behavior of spline curves

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Abstract

Shape preservation behavior of a spline consists of criterial conditions for preserving convexity, inflection, collinearity, torsion and coplanarity shapes of data polgonal arc. We present our results which acts as an improvement in the definitions of and provide geometrical insight into each of the above shape preservation criteria. We also investigate the effect of various results from the literature on various shape preservation criteria. These results have not been earlier refered in the context of shape preservation behaviour of splines. We point out that each curve segment need to satisfy more than one shape preservation criteria. We investigate the conflict between different shape preservation criteria 1)on each curve segment and 2)of adjacent curve segments. We derive simplified formula for shape preservation criteria for cubic curve segments. We study the shape preservation behavior of cubic Catmull-Rom splines and see that, though being very simple spline curve, it indeed satisfy all the shape preservation criteria.

1 Introduction

Designers in industries need to create splines which can interpolate the data points in such a way that they preserve the shape of polygonal arc formed by data points. Among the properties that the spline curves need to satisfy following properties are of common interest to almost all the designers:

- Smoothness

- Preservation of shape of the data polygon

- Each curve segment to be a low order polynomial curve.

We first illustrate the shape preservation behaviour of a spline interpolating planar data points with the help of figures 1, 2 and 3.

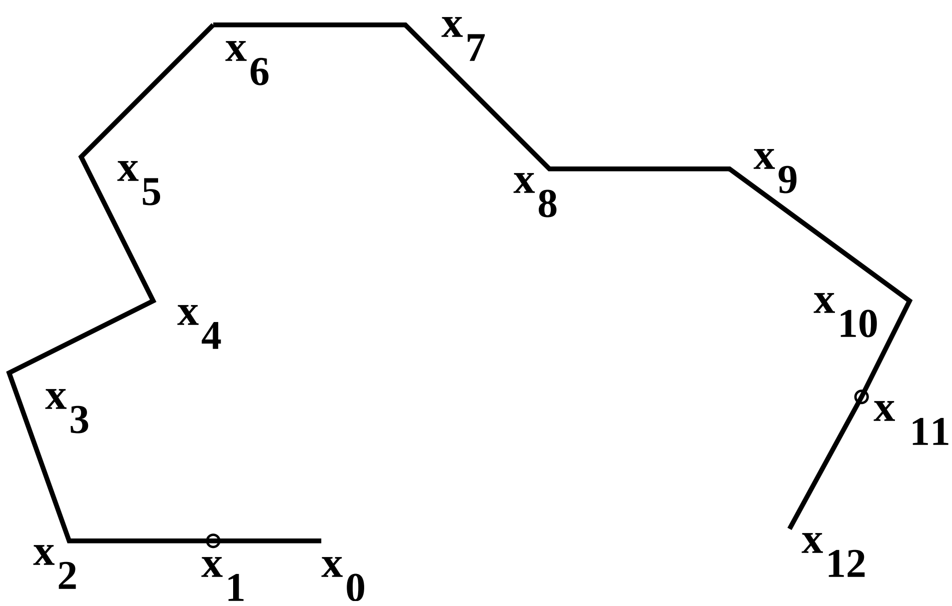


Figure 1: Data points to be interpolated

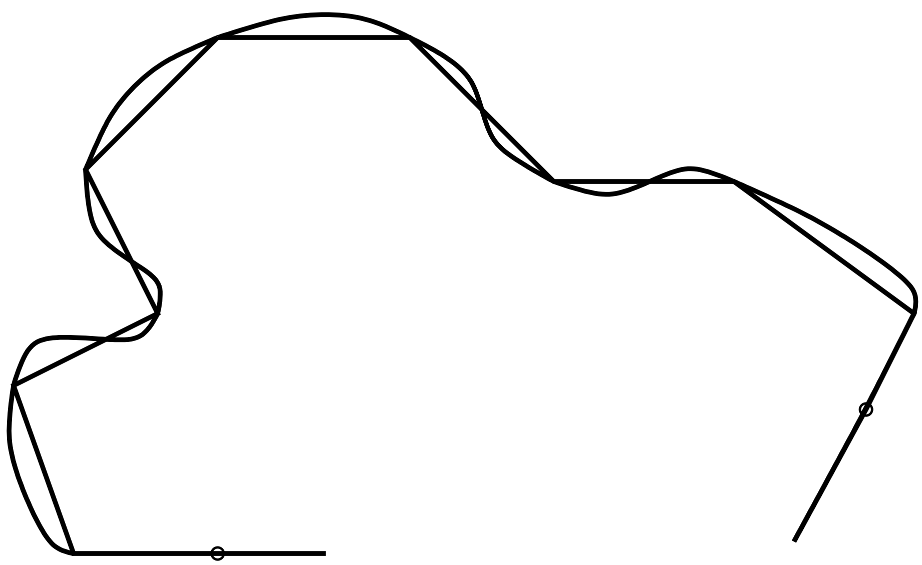


Figure 2: Data polygon with a shape preserving interpolating spline

One can observe that

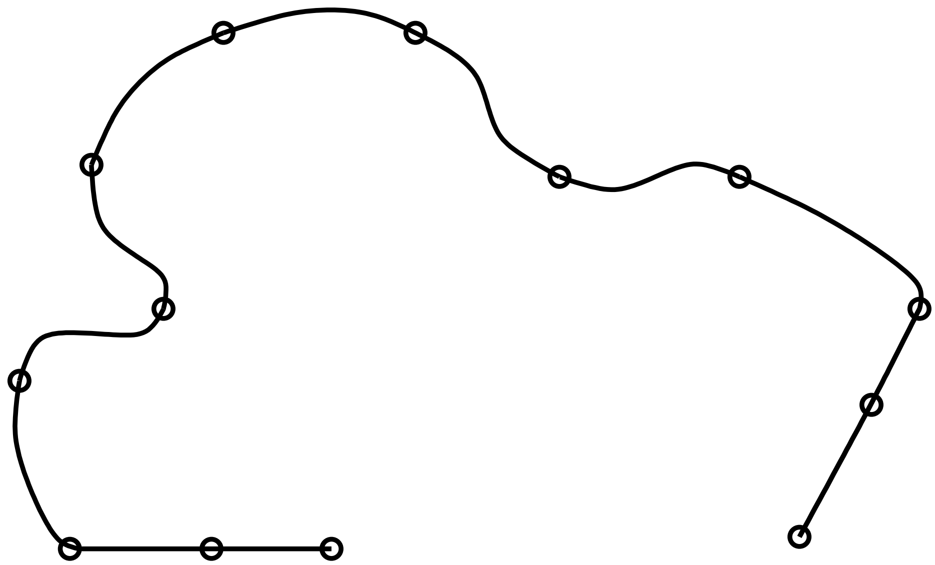


Figure 3: Data points with shape preserving interpolating spline

- Inflection depicted by data points $\mathbf{x}_3$ $\mathbf{x}_4$ $\mathbf{x}_5$ and $\mathbf{x}_6$ is preserved by suitable inflection of the curve segment between $\mathbf{x}_4$ and $\mathbf{x}_5$
- Convexity depicted by data points $\mathbf{x}_4$ $\mathbf{x}_5$ $\mathbf{x}_6$ and $\mathbf{x}_7$ is preserved by the convex shape of the curve segment between $\mathbf{x}_5$ and $\mathbf{x}_6$
- Collinearity depicted by data points $\mathbf{x}_0$ $\mathbf{x}_1$ and $\mathbf{x}_2$ is preserved by collinearity of the curve segment between $\mathbf{x}_0$ and $\mathbf{x}_2$

an so on. It can also be observed that shape preserving behaviour of a spline makes it more close to mimicing free-hand curve drawing.

However, modelling of shape preservation behaviour of interpolating splines in R^3 is relatively difficult. Typically shape preservation criteria that have been studied for the generation of interpolating splines consists of conditions for the preservation of 1)Convexity 2)Inflection 3)Collinearity 4)Torsion and 5)Coplanarity shapes of data polygonal arc (formed by line joining the consecutive points of ordered set of data points). In this paper we present literature survey and also our analysis for each of the above criteria. In sections 3.1 and 3 we present our analysis for convexity preservation criteria. In literature, basically, two definitions for convexity preservation criteria are followed. One which is followed in [8, Kaklis and Karavelas, 1997], [10, 12, Costantini et. al.] etc. and another which is followed in [13, Kong and Ong, 2002], [7, Goodman and Ong, 1997] etc. The latter definition includes the conditions of previous definition and hence we investigate further on the later definition. We find that in the later definition,

for a large set of interpolating spline curve (containing set of rational spline, excluding straight lines, curves as small subset) one of the two conditions is redundant (that is, the condition is actually taken care by other condition of the definition) for all the points of the curve except for set of points on the curve of measure zero (the set is finite for rational curves excluding straight lines). Also at these points the error (if at all occurs) is very negligible and it is observed that in all almost all the algorithm staright line segments in the spline curves are considered for collinearity preservation criteria. We use our lemma 3.13 and the characterization of convexity of planar curve presented in [6, Liu and Traas, 1997] for our analysis of convexity criteria of interpolating splines. In [6, Liu and Traas, 1997] the characterization of convexity of planar curve on R^2, that is, XY plane have been derived. We use this characterization to get the characterization of planar curve on any plane in R^3. Then the modified characterization has been used to improve the definition of convexity preservation criteria of splines. We further state simplified characterization of convexity preservation criteria for cubic splines in terms of control polygon of individual Bézier segments.

In sections 4 to 6 we present the analysis for inflection preservation criteria of splines. We refer two papers [5, Goodman, 1991] and [7, Goodman and Ong, 1997] for our analysis for inflection criteria. In [5, Goodman, 1991] author has stated definitions and results for inflection counts for planar and space curves and polygonal arcs. In [7, Goodman and Ong, 1997] authors have defined the inflection criteria of splines and have constructed a spline satisfying the criteria. However, in [7, Goodman and Ong, 1997], authors haven't indicated any connection with the analysis of [5, Goodman, 1991]. Also the analysis in [5, Goodman, 1991] is not used in [7, Goodman and Ong, 1997] for the analysis of inflection criteria of splines. We observe in section 6 that our lemma 3.13 acts as a connection between the analysis in sections 4 and 5 and conditions stated in definition of convexity criteria of splines.

In section 4 we state definitions and results from [5, Goodman, 1991] for inflection count for planar curves and polygonal arcs. The relation between the inflection counts of planar B-spline and Bézier curves and inflection counts of their control polygons are stated. In section 5 we state definitions and results for inflection counts for space curves and polygonal arcs from [5, Goodman, 1991]. Definitions in section 5 uses the definitions in section 4. In section 6 we state the definition and analysis for the convexity criteria for splines. In section 7 we state the results from [5, Goodman, 1991] which states (via lemma 3.13) the conditions under which a spline curve does not satisfy inflection criteria or convexity criteria.

In sections 8, 9 and 10 we analyze and give improved conditons for collinearity , torsion and coplanarity preservation criteria respectively for splines. In section 11 and 12 we analyze and give condtions on a spline curve to resolve conflict between different shape preservation conditions on a curve segment or a pair of adjacent curve segments.

Almost every geometric modeler necessarily uses cubic splines for generating 3D models of products. Cubic splines are computationally most viable solution for various applications requiring complicated geometric operations. They are also the splines of least degree that can exhibit torsion. Thus we see that it is necessary that cubic splines should preserve shape of the data points they interpolate. In section 13 we describe our analysis and results for shape preserving criteria for cubic splines. The conditions for shape preserving criteria for cubic splines derived in this section are expressed in terms data points and slopes at data points. During our analysis we obtain simplified formula for discrete shape measures and a new property for Bézier curves.

In subsection 13.1 we state aome additional notations required in section 13. In subsections 13.2, 13.3, 13.4, 13.5 and 13.6 we derive the conditions for convexity, inflection, torsion, collinearity and coplanarity preservation criteria respectively, for the cubic curve segment in terms of the two data points at their ends and slope vectors at them. In subsection 13.4 we simplify the expression for torsion of cubic Bézier curves. In subsection 13.5 we derive an expression for sine of the angle between a point vector on a Bézier curve with a given vector and state it in theorem 13.17 and we use it to get simplified formula for collinearity preservation criteria for cubic curve segments. Further we use this analysis in subsection 13.6 to get simplified formula for coplanrity preservation criteria for cubic curve segments. In section 14 we investigate shape preservation behavior of cubic Catmull-Rom splines. Finally in section 15 we state our conclusions about the analysis in this paper.

2 Notations and prelimnaries

Let $\mathbf{x}_i \in R^3$, $i = 0, ..., n$ be $n + 1$ data points and $\mathcal{D}$ be the polyline or polygonal arc formed by joining the points with each side being $L_i = \mathbf{x}_i - \mathbf{x}_{i-1}$, $i = 1, ..., n$. Let $N_i = L_{i-1} \times L_i$. In discrete differential geometry [1, Sauer, 1970] discrete binormal is defined as $\dfrac{N_i}{|N_i|}$. For a curve $\gamma(t) = [x(t), y(t), z(t)]$, $t \in [0, 1]$ in R^3 let $\omega(t) = \gamma'(t) \times \gamma''(t)$.

3 Convexity preservation criteria for interpolating splines

In [9, Karvelas and Kaklis, 2000], [8, Kaklis and Karavelas, 1997] [10, Costantini, Goodman, Manni, 2000] [12, Costantini, Cravero, Manni, 2002], [15, Manni, Pelosi, 2004] we have the following definition

Definition 3.1 *Convexity preservation criteria for a curve $\gamma(t)$ interpolating data points consists of following condition:*

1. *if $N_{i-1} \cdot N_i > 0$, then $\omega(t) \cdot N_m > 0$, $t \in [t_{i-1}, t_i]$, $m = i-1, i$.*

The above definition 3.1 takes into account the ability of a spline to appear as convex curve along only two viewpoints N_{i-1} and N_i.

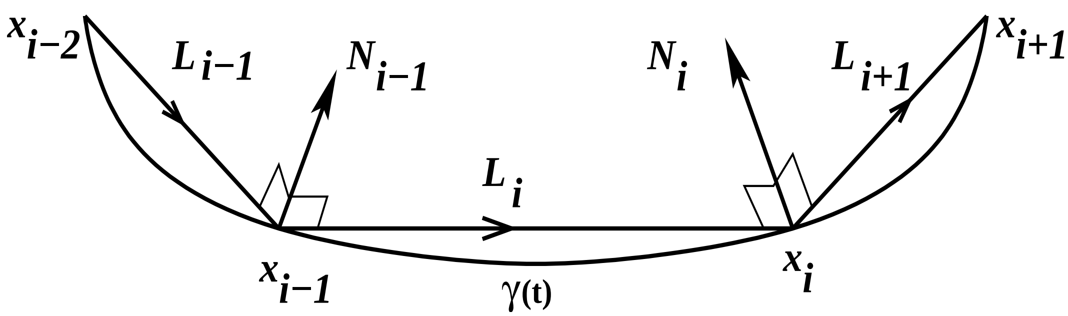


Figure 4: Data point with $N_{i-1} \cdot N_i > 0$ requiring convexity preservation by i^{th} curve segment

The following definition is from [7, Goodman and Ong, 1997], [13, Kong and Ong, 2002].

Definition 3.2 *[7, Goodman and Ong, 1997] Convexity preservation criteria for a curve $\gamma(t)$ interpolating data points consists of following condition:*

1. *If $N_{i-1} \cdot N_i > 0$, then for all $N = \lambda N_{i-1} + \mu N_i$, where $\lambda, \mu \geq 0$, $(\lambda, \mu) \neq (0,0)$, the projection $P_{N^\perp}\gamma(t)$, $t \in [t_{i-1}, t_i]$, is globally convex and*

2. *$\omega(t) \cdot N > 0$.*

Now we state a theorem from [7, Goodman and Ong, 1997] which makes the convexity condition of projection curves simpler.

Theorem 3.3 *[7, Goodman and Ong, 1997] Suppose that R is a curve in R^3 and V_1 and V_2 are vectors in R^3 so that $P_{V_1^\perp}R$ and $P_{V_2^\perp}R$ are convex with the same orientation with respect to V_1 and V_2 respectively. Then $P_{V^\perp}R$ is convex with the same orientation with respect to V where $V = \lambda V_1 + \mu V_2$ for any $\lambda, \mu \geq 0$, $(\lambda, \mu) \neq (0,0)$.*

Thus from the theorem (3.3) we can observe that condition requiring $P_{N_{i-1}^{\perp}}\gamma_i(t)$ and $P_{N_i^{\perp}}\gamma_i(t)$ to be convex is equivalent to the condition requiring curves $P_{N^{\perp}}\gamma_i(t)$ for $N = \lambda N_{i-1} + \mu N_i$ for any $\lambda, \mu \geq 0$, $(\lambda, \mu) \neq (0, 0)$, to be convex.

3.1 Convexity of planar curves

3.1.1 Convexity of planar curves on XY plane

In [6, Liu, Traas, 1997] authors have defined local and global convexity of a planar curve as follows. The author distiguishes convexity and concavity of planar curves in terms of the orientation we assign to the curve. Consider a curve $\gamma(t) = (x(t), y(t))$, $t \in [0, 1]$ in R^2. An oriented planar curve is an ordered set in R^2, given by $\gamma(t) = [x(t), y(t)]$, $t \in [0, 1]$ with direction from $t = 0$ to $t = 1$. A global supporting line of an oriented curve $\gamma(t)$ at a point $\gamma(t_0)$ is an oriented line, L, having consistent direction with $\gamma(t)$ in t_0, and satisfying (a) $\gamma(t_0)$ is a point of L; (b) the entire curve $\gamma(t)$, $t \in [0, 1]$, lies in one closed half-plane with respect to L.

A global supporting line of an oriented curve $\gamma(t)$ at a point $\gamma(t_0)$ is an oriented line, L, having consistent direction with $\gamma(t)$ in t_0, and satisfying

1. $\gamma(t_0)$ is a point of L;

2. The entire curve $\gamma(t)$, $t \in [0, 1]$, lies in one closed half-plane with respect to L.

A local supporting line of an oriented curve $\gamma(t)$ at a point $\gamma(t_0)$ is an oriented line, L, having consistent direction with $\gamma(t)$ in t_0, and satisfying

1. $\gamma(t_0)$ is a point of L;

2. A local neighborhood $\gamma(t)$, $t \in [t_1, t_2]$, of $\gamma(t_0)$, lies in one closed half-plane with respect to L, where t_1 and t_2 satisfy

$$0 \leq t_1 < t_0 < t_2 \leq 1 \text{ or } 0 = t_1 = t_0 < t_2 \leq 1 \text{ or } 0 \leq t_1 < t_0 = t_2 = 1. \tag{3.1}$$

Definition 3.4 *[6, Liu, Traas, 1997] $\gamma(t)$, $t \in [0, 1]$, is a globally convex curve if it satisfy:*

1. *There is at-least one global supporting line at every point of $\gamma(t)$;*

2. *The entire curve lies in the right closed half-plane with the supporting line as its left boundary.*

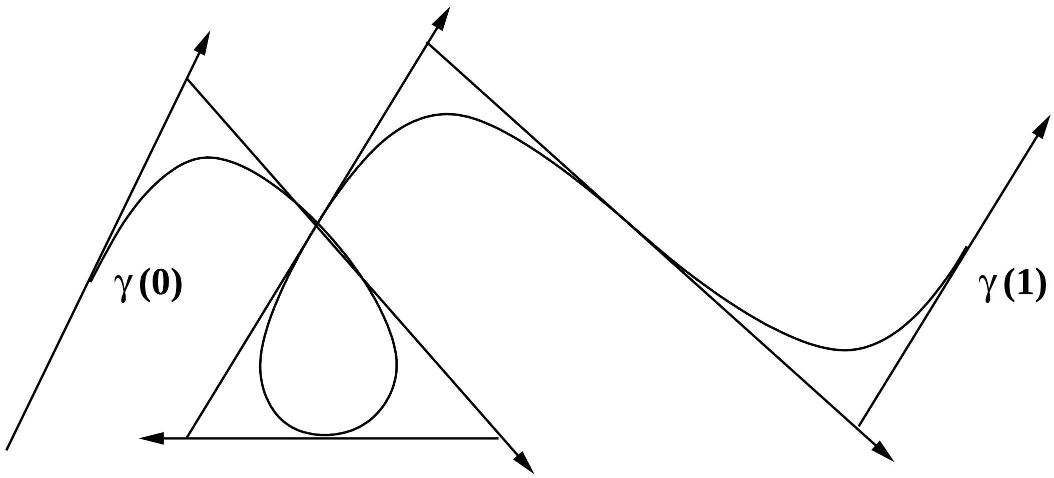
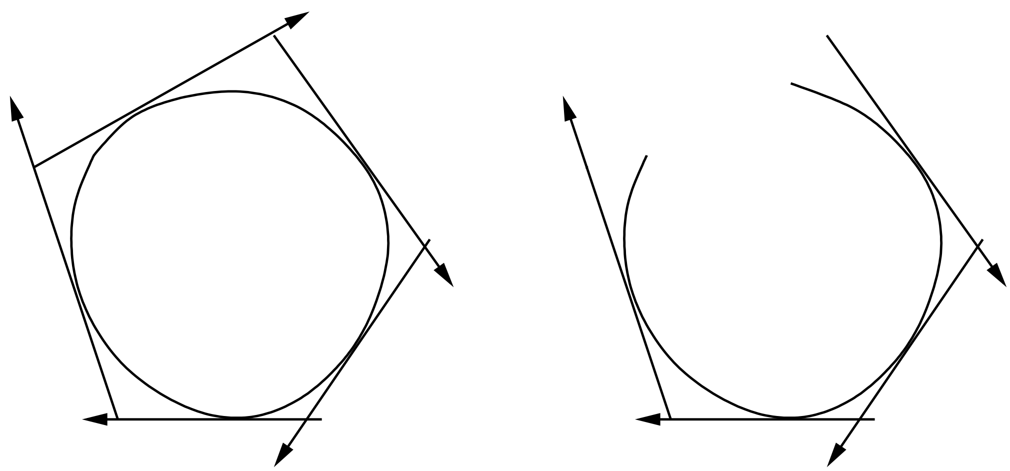


Figure 5: Supporting and non-supporting lines [Liu and Traas '97]



(a) closed

(b) open

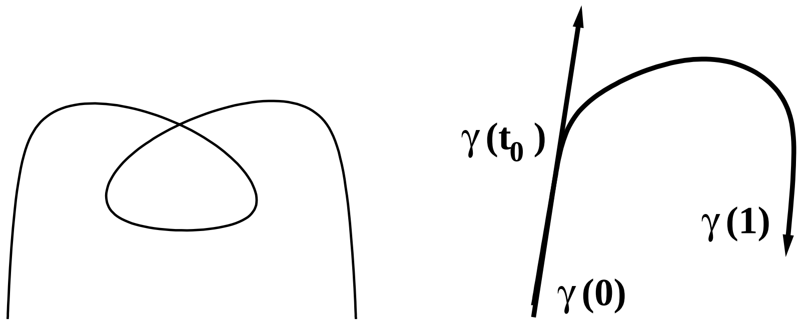
Figure 6: Globally convex curve [Liu and Traas '97]

Definition 3.5 *[6, Liu, Traas, 1997] $\gamma(t)$, $t \in [0,1]$, in definition (3.4) is a locally convex curve if in (1) the global supporting line is replaced by a local one and (2) is true only for a related local neighborhood.*

A globally and locally concave curve has the same definition as for globally and locally convex curve respectively with left and right interchanged.

We observe that the above definitions [1] hold for any planar curve in R^3. However, the two main theorems of [6, Liu and Traas, 1997] that we state below need some modifications.

[1] There are other definitions of a convex curve. For example, in [4, Farin] it is defined as a part of boundary of a convex set. In [6, Liu and Traas, 1997] is has been proved, using Hahn-Banach theorem, that this definition is included in definition (3.4).



(a) (b)

Figure 7: (a) Locally convex curve [Liu and Traas '97] (b) Condition 3.6 for global convexity [Liu and Traas '97]

Theorem 3.6 *[6, Liu and Traas, 1997] $\gamma(t)$, $t \in [0,1]$, is locally convex if and only if*

$$\gamma'(t) \times \gamma''(t) \leq 0, t \in [0,1] \tag{3.2}$$

where $\gamma(t)$ is C^2-continuous, $\gamma'(t)$, $\gamma''(t)$ are first and second derivatives of $\gamma(t)$, and

$$\gamma'(t) \times \gamma''(t) = \begin{vmatrix} x'(t) & x''(t) \\ y'(t) & y''(t) \end{vmatrix} = x'(t)y''(t) - x''(t)y'(t). \tag{3.3}$$

∎

Theorem 3.7 *[6, Liu and Traas, 1997] A curve $\gamma(t)$ satisfying the condition*

$$\gamma(t) \neq \gamma(0) \text{ for } t \in (0,1) \tag{3.4}$$

is globally convex if and only if

$$\gamma'(t) \times \gamma''(t) \leq 0, \tag{3.5}$$

$$(\gamma(t) - \gamma(0)) \times \gamma'(t) \leq 0, t \in [0,1], \tag{3.6}$$

$$\gamma'(0) \times (\gamma(t) - \gamma(0)) \leq 0, t \in [0,1]. \tag{3.7}$$

∎

Remark 3.8 *For a curve $\gamma(t) \in R^2$, $\gamma'(t) \times \gamma''(t)$ is its curvature. Therefore, in 3.2 and 3.5 the equality holds at $t = t_1$ if and only if $\gamma(t)$ behaves locally as straight line (turns with angle 0^o) at $t = t_1$. One can replace 3.2 and 3.5 by $\gamma'(t) \times \gamma''(t) < 0$ by requiring strict convexity of $\gamma(t)$. But we did not find this to be significant requirement as set of points for which equality*

holds is of measure zero for almost all curves (except straight lines) used for interpolation (and therefore their local behaviour as straight line have negligible effect). The conditions 3.6 and 3.7 ensures that the curve doesn't intersect itself and equality in these conditions may cause the curve to become straight line in some cases.

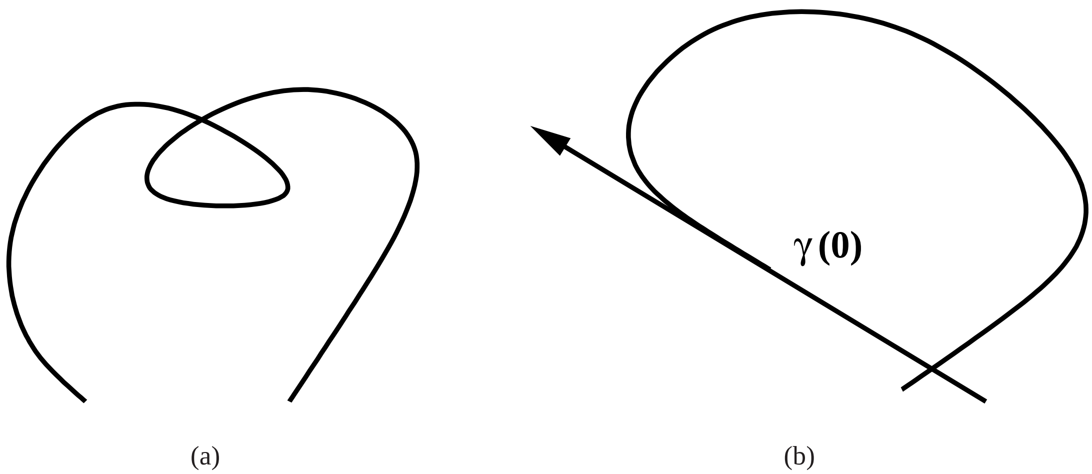


Figure 8: Non-convex curve (a) without 3.6 (b) without 3.7 [Liu and Traas '97]

3.1.2 Convexity of planar curves in R^3

In this subsection we extend the characterization of globally convexity of curve in R^2 to the characterization of global convexity of planar curve $\gamma(t) = [x(t), y(t), z(t)]$, $t \in [0, 1]$ in R^3. In the remaining part of this chapter, except in section 4, we use the operator $\times$ to denote the cross product of two vectors, that is,

$$A \times B = \begin{vmatrix} i & j & k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \tag{3.8}$$

where $A = [A_x, A_y, A_z]$, $B = [B_x, B_y, B_z] \in R^3$, i, j and k are unit vectors along x, y and z axis respectively. We observe that [6, Liu and Traas, 1997] the operator $\times$ (in R^2) is mainly used to understand the direction in which the curve bends with respect to the orientation induced by $z-$axis. We now explain our characterization for global convexity of planar curve in R^3 which solves our above purpose as follows.

It has been observed in [6, Liu and Traas, 1997], the convexity and concavity depends on its orientation, that is, direction in which is traversed. By inverting the orientation, a convex curve

turns into a concave curve and vice versa. Definition based on direction helps to distinguish between two curves as convex and concave curves in most of the practical situations. For example, in the case of two curves $\gamma(t)$ and $Q(t)$ ($t \in [0,1]$) such that $\gamma(1)$ and $Q(1)$ lie on the right side of $\gamma(0)$ and $Q(0)$ respectively.

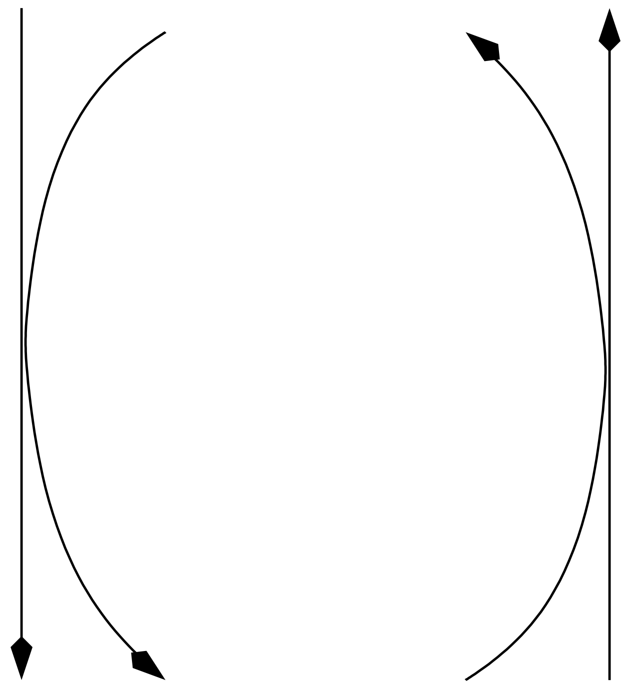


Figure 9: Convex curves with different orientations [Liu and Traas '97]

But the direction of orientation itself gets inverted if the normal to the plane is taken to be the normal with its direction opposite to the given normal. (Note that in (3.3) "$\times$" can be interpreted as to denote the dot product of the cross product between two vectors in $x-y$ plane with unit vector along z-axis (which is along normal dierction to the $x-y$ plane)). In our case we need the convexity of the curve according to orientation induced by a specified normal vector as it requires that the spline curve to be convex in the same direction as the (data) polygonal arc. The normal vector is specified as the normal to the plane containing a pair of adjacent line segments in the data polygonal arc. (Therefore we do not consider the case of concavity.)

Thus we have the following definition for local and global convexity of a planar curve lying on a plane Π in R^3 with respect to orientation of normal vector N of the plane Π.

Definition 3.9 *A C^2-continuous planar curve $\gamma(t)$, $t \in [0,1]$, is locally convex if and only if* $(\gamma'(t) \times \gamma''(t)) \cdot N \geq 0, t \in [0,1]$.

Definition 3.10 *A C^2-continuous planar curve $\gamma(t)$, $t \in [0,1]$, satisfying the condition*

$$\gamma(t) \neq \gamma(0) \text{ for } t \in (0,1) \tag{3.9}$$

is globally convex if and only if it satisfies following conditions.

1. $(\gamma'(t) \times \gamma''(t)) \cdot N \geq 0$
2. $((\gamma(t) - \gamma(0)) \times \gamma'(t)) \cdot N \geq 0$, $t \in [0,1]$
3. $(\gamma'(0) \times (\gamma(t) - \gamma(0))) \cdot N \geq 0$, $t \in [0,1]$

Following remarks about defintions 3.9 and 3.10 are very significant.

Remark 3.11 *Since $\gamma'(t)$, $\gamma''(t)$ and $(\gamma(t) - \gamma(0))$, $t \in [0,1]$ are parallel to the plane Π. Therefore sign of dot product of each cross product with N actually represents the bending of the curve according to the orientation induced by N.*

Remark 3.12 *Remark 3.8 about equality holds for the inequalities appearing in the definitions 3.9 and 3.10 holds.*

3.2 Improvement in the condition of convexity preservation criteria

Now using our lemma, stated below, we find that for almost all interpolating splines the condition $\omega(t) \cdot N > 0$ in definition 3.2 is implied by global convexity of $P_{N^\perp}(\gamma(t))$ (and need not be stated separately) for almost all values of t. Thus we further simplify definition of convexity criteria. This lemma also helps to modify the definition for inflection criteria to get a simpler definition in section 6.

Lemma 3.13 *Let $P_{N^\perp}(\gamma(t))$ be denoted as $\gamma_N(t)$, $\omega(t) = \gamma'(t) \times \gamma''(t)$ and $\omega_N(t) = \gamma'_N(t) \times \gamma''_N(t)$. Then $\omega_N(t) \cdot N = \omega(t) \cdot N$.*

Proof: A plane with normal vector N is given by $\dfrac{(x,y,z) \cdot N + d}{\|N\|} = 0$, $d \in R$. We know that $\gamma_N(t) = \gamma(t) + \dfrac{\gamma(t) \cdot N + d}{\|N\|^2} N$. Therefore $\gamma'_N(t) = \gamma'(t) + \dfrac{\gamma'(t) \cdot N + d}{\|N\|^2} N$,
$\gamma''_N(t) = \gamma''(t) + \dfrac{\gamma''(t) \cdot N + d}{\|N\|^2} N$. From the above we get
$\omega_N(t) = (\gamma'(t) \times \gamma''(t)) + (\gamma'(t) \times \dfrac{\gamma''(t) \cdot N + d}{\|N\|^2} N) + (\dfrac{\gamma'(t) \cdot N + d}{\|N\|^2} N \times \gamma''(t))$. Thus
$\omega_N(t) \cdot N =$

$$(\gamma'(t)\times\gamma''(t))\cdot N+(\gamma'(t)\times\frac{\gamma''(t)\cdot N+d}{\|N\|^2}N)\cdot N+(\frac{\gamma'(t)\cdot N+d}{\|N\|^2}N\times\gamma''(t))\cdot N=\omega(t)\cdot N.$$

Hence proved. ∎

Now we first analyse the condition $\omega(t)\cdot N\geq 0$.

- For a large class of curves, including rational curves (except for straight lines) as a small subset, $\omega(t)\cdot N=0$ holds for a set of values of $t\in[0,1]$ whose measure is zero.
- $\omega(t)\cdot N=0$ at $t=t_1$ if and only if $|\omega_N(t)|=0$ that is, $\gamma_N(t)$ behaves as straight line at $t=t_1$. Such behavior at $t=t_1$ and not in its neighbbourhood, has negligible effect on shape of the projected curve (along a viewpoint).
- For some cases $\omega(t)\cdot N=0$ at $t=t_1$ may also imply that $\omega(t)$, which is binormal of the curve $\gamma(t)$, is perpendicular to N at $t=t_1$. That is, N is parallel to the osculating plane (plane on which curve lies locally) of $\gamma(t)$ at $t=t_1$. This might not be good unless such a torsion is required. Also, this may occur to great extent even for the case where $\epsilon>\omega(t)\cdot N>0$, for sufficiently small values of ϵ. However, possibility of such torsion, if undesired, can be controlled by other shape preservation criteria like torsion, coplanarity and collinearity preservation criteria.

From the above analysis and lemma 3.13 we get following two observations. First, from definition 3.9 (for local convexity of a curve), we see that the definition 3.1 requires the projection of curve on the plane perpendicular to N_{i-1} and N_i be only locally convex which one can observe from figure 8(a) that the spline may not always serve the purpose of shape preservation.

Second, according to the definition 3.10 for convexity of a planar curve in R^3 one of the condition that $\gamma_N(t)$ need to satisfy, to be globally convex, is $\omega_N(t)\cdot N\geq 0$. From our lemma 3.13, global convexity of $P_{N^\perp}\gamma(t)$ implies $\omega(t)\cdot N>0$. Thus we see that the condition $\omega(t)\cdot N>0$ is redundant in the definition 3.2.

We now state our results which will further simplify the conditions in the definition 3.2, as lemmas. The proofs of these lemmas are similar to that of lemma 3.13

Lemma 3.14 *With the notation same as that in lemma 3.13 we have*

$$((\gamma_{N^\perp}(t)-\gamma_{N^\perp}(0))\times\gamma'_{N^\perp}(t))\cdot N \;=\; ((\gamma(t)-\gamma(0))\times\gamma'(t))\cdot N$$

Lemma 3.15 *With the notation same as that in lemma 3.13 we have*

$$(\gamma'_{N^\perp}(0)\times(\gamma_{N^\perp}(t)-\gamma_{N^\perp}(0)))\cdot N \;=\; (\gamma(0)\times(\gamma(t)-\gamma(0)))\cdot N$$

Using theorem 3.3 and lemmas 3.13, 3.14 and 3.15, we have the following improved definition for convexity criteria for splines.

Definition 3.16 *A spline curve $\gamma(t)$ interpolating data points satisfies convexity criteria if, for $j = i-1, i$,*

1. $\omega(t) \cdot N_j \geq 0$,

2. $((\gamma(t) - \gamma(0)) \times \gamma'(t)) \cdot N_j \geq 0$,

3. $(\gamma'(0) \times (\gamma(t) - \gamma(0))) \cdot N_j \geq 0$,

$t \in [t_{i-1}, t_i]$, *whenever* $N_{i-1} \cdot N_i > 0$.

4 Inflection of a planar curves and polygonal arcs

In [5, Goodman, 1991] authors have given definitions and conditions for the existence of inflections in a planar curve as follows. In this section for $A = (A_1, A_2)$, $B = (B_1, B_2)$ in R^2 we write

$$A \times B \;=\; A_1 B_2 - A_2 B_1 \tag{4.1}$$

For any sequence $a = (a_1, \ldots, a_n)$ in R^n, we define $S(a) = S(a_1, \ldots, a_n)$ to be the number of strict sign changes in the sequence.

Definition 4.1 *[5, Goodman, 1991] We say a polygonal arc $P_0 P_1 \ldots P_n$ for points P_0, P_1, ..., P_n in R^2 is regular if the following hold*

1. *It turns through a total angle of magnitude at most π, that is, for some V in R^2, $V \cdot (P_i - P_{i-1}) \geq 0$, $i = 1, \ldots, n$.*

2. *It does not turn through an angle of π at any vertex, that is, for any $0 < i \leq j < n$ with $P_{i-1} \neq P_i = P_{i+1} \ldots = P_j \neq P_{j+1}$, $P_i - P_{i-1} \neq \lambda(P_{j+1} - P_j)$ for any $\lambda < 0$.*

Definition 4.2 *[5, Goodman, 1991] For a regular polygonal arc $P_0 P_1 \cdots P_n$ in R^2, with the condition $P_{i-1} \neq P_i$, $i = 1, \ldots, n$ denoted by Γ we define inflection count as*

$$i(\Gamma) \;=\; S(V_1, \ldots, V_{n-1}) \tag{4.2}$$

where

$$V_i \;=\; (P_i - P_{i-1}) \times (P_{i+1} - P_i) \; i = 1, \ldots, n-1. \tag{4.3}$$

For any function $f : (a, b) \rightarrow R$ we define $S(f)$ to be the number of strict sign changes in $f(t)$, $a < t < b$, that is $S(f) = supS(f(t_1), \ldots, f(t_n))$, where the supremum is taken over all sequences $a < t_1 < \ldots < t_n = b$, for all n. For a curve $\gamma(t) \in R^2$, $t \in [a, b]$ which is constant for t in $[\alpha, \beta] \subset (a, b)$ but not on any larger interval, we define

$$K(t) = \begin{cases} \frac{1}{2}\{u(\alpha^-) \times u'(\alpha^-) + u(\beta^-) \times u'(\beta^-)\}, & \text{if } u(\alpha^-) = u(\beta^+), \\ u(\alpha^-) \times u(\beta^+), & \text{if } u(\alpha^-) \neq u(\beta^+). \end{cases} \tag{4.4}$$

Definition 4.3 *Inflection count for a curve* $\gamma(t) \in R^2$, $t \in [a, b]$, *is defined as* $i(\gamma) := S(K)$. *It is actually the number of times the curve changes from turning in a clockwise direction to turning in an anti-clockwise direction, or vice-versa.*

With the above definitions regarding inflection count we state the following relation between inflection count of B-spline curve and its control polygon from [5, Goodman, 1991].

Theorem 4.4 *[5, Goodman, 1991]Suppose*

$$\gamma(t) \;=\; \sum_{i=-n}^{m-1} A_i N_i(t), t_0 \leq t \leq t_m, \tag{4.5}$$

and Γ *denotes the polygonal arc* $A_{-n}A_{-n+1}\cdots A_{m-1}$. *If* $A_{i-n}\cdots A_i$ *is regular for* $i = 0, \ldots, m-1$, *then*

$$i(\gamma) \;\leq\; i(\Gamma). \tag{4.6}$$

where N_i *denote B-spline basis function.*

We now state the relation from [5, Goodman 1991] between inflection count of Bézier curve and its control polygon which follows as a corollary to the above theorem.

Corollary 4.5 *[5, Goodman, 1991] If*

$$\gamma(t) \;=\; \sum_{i=1}^{n} A_i \tbinom{n}{i} t^i (1-t)^{n-i}, 0 \leq t \leq 1, \tag{4.7}$$

and the polygonal arc $\Gamma = A_0A_1\cdots A_n$ *is regular, then*

$$i(\gamma) \leq i(\Gamma). \tag{4.8}$$

Now we state a theorem from [5, Goodman, 1991] which says that the above result does not hold if the control polygon is not regular.

Theorem 4.6 *[5, Goodman 1991] Let $\gamma(t)$ be cubic Bézier curve given by*

$$\gamma(t) \;=\; A(1-t)^3 + 3Bt(1-t)^2 + 3Ct^2(1-t) + Dt^3 \tag{4.9}$$

where A, B, C, $D \in R^2$ are control points and Γ denote the polygonal arc ABCD. Suppose Γ turns through an angle of magnitude $> \pi$ and let P be the point of intersection of the line through A and B and the line through C and D. Then

$$i(\gamma) \;=\; \begin{cases} 0, \textit{ if} |B-A||C-D|/|B-P||C-P| \leq 4, \\ 2, \textit{ if} |B-A||C-D|/|B-P||C-P| > 4. \end{cases} \tag{4.10}$$

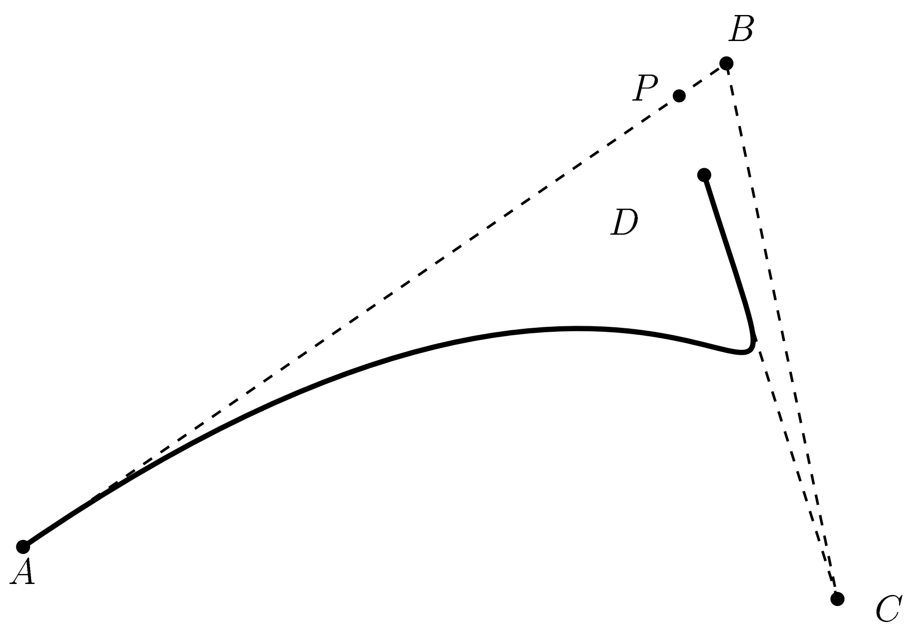


Figure 10: Cubic Bézier curve with two inflection points whose control polygon has no inflection

5 Inflections of curves and polygonal arcs in R^3

Let $\gamma : [a,b] \to R^3$ be a curve in R^3. We also resume the meaning of operator $\times$ as cross product of two vectors in R^3 as defined in subsection 4.1.3.

For any w in $S_1 = \{v \in R^3 : |v| = 1\}$ we shall denote by P_w the orthogonal projection from R^3 onto the 2-dimensional subspace orthogonal to w, that is, $P_w x = x - (x \cdot w)w$.

Definition 5.1 *[5, Goodman, 1991] Inflection count $I(\gamma)$ of the (spatial) curve γ to be the maximum number of inflections that be seen in γ by observing from any direction, that is,*

$$I(\gamma) \;=\; \textit{ess sup}\{i(P_w\gamma) : w \in S_1\}. \tag{5.1}$$

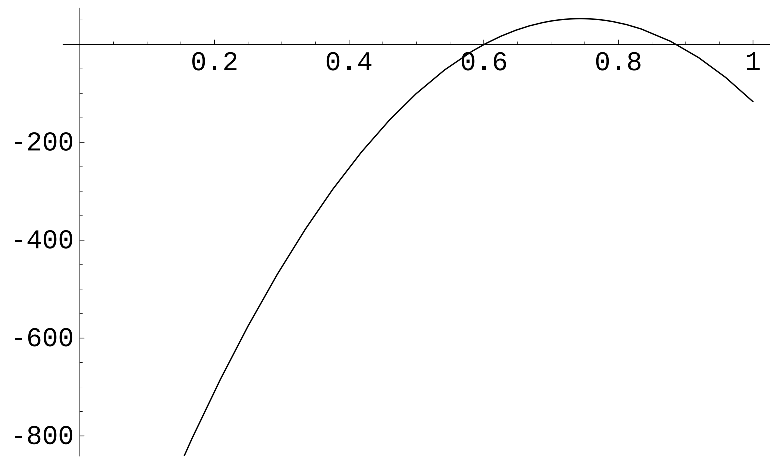


Figure 11: Curvature plot of the curve in figure 10

We suppose that γ is continuous with piecewise C^1 unit tangent vector $u(t) = \gamma'(t)/|\gamma'(t)|$. As before suppose $\gamma(t)$ is constant for t in $[\alpha, \beta] \subset (a, b)$ but not on any larger interval and we define for $\alpha \leq t \leq \beta$,

$$K(t) \quad = \quad \begin{cases} \frac{1}{2}\{u(\alpha^-) \times u'(\alpha^-) + u(\beta^+ \times u'(\beta^+)), \text{ if } u(\alpha^-) = u(\beta^+), \\ u(\alpha^-) \times u(\beta^+), \text{ if } u(\alpha^-) \neq u(\beta^+). \end{cases} \tag{5.2}$$

Theorem 5.2 *[5, Goodman, 1991] Suppose $\gamma : [a, b] \rightarrow R^3$ is continuous with piecewise C^1 unit tangent vector $u(t) = \frac{\gamma'(t)}{|\gamma'(t)|}$. Then*

$$I(\gamma) \quad = \quad sup\ \{S(w.K) : w \in S_1\} \tag{5.3}$$

Corollary 5.3 *If the curve γ, as in theorem (5.2), lies in a plane with a normal n, then*

$$I(\gamma) \quad = \quad S(n \cdot K). \tag{5.4}$$

Theorem (5.2), also implies the following definition for polygonal arc.

Definition 5.4 *[5, Goodman, 1991] For a polygonal arc $P_0P_1 \cdots P_n$ and $P_{i-1} \neq P_i$, $i = 1, ..., n$, denoted by Γ then its inflection count is*

$$I(\Gamma) \quad = \quad sup\ \{S(w \cdot V_1, \ldots, w \cdot V_n) : w \in S_1\}. \tag{5.5}$$

6 Inflection preservation criteria for interpolating splines

Let $\mathbf{x}_i \in R^3$, $i = 0, ..., n$ be $n+1$ data points and $\mathcal{D}$ be the polyline joining the points with each side being $\mathbf{L}_i = \mathbf{x}_i - \mathbf{x}_{i-1}$, $i = 1, ..., n$. Let $N_i = L_{i-1} \times L_i$. In discrete differential geometry

[Sauer, 1970] discrete binormal is defined as $\dfrac{N_i}{|N_i|}$. For a curve $P(t) = [x(t), y(t), z(t)]$, $t \in [0,1]$ in R^3 let $\omega(t) = P'(t) \times P''(t)$.

In [10, Costantini, Goodman, Manni, 2000], [12, Costantini Cravero Manni, 2002] [15, Manni, Pelosi, 2004] we have the following definition

Definition 6.1 *[10, Costantini, Goodman, Manni, 2000] A curve $\gamma(t)$ interpolating data points satisfies inflection criteria if is satisfy the condition that, if $N_{i-1} \cdot N_i < 0$, then $(\omega(t_l) \cdot N_m)(N_l \cdot N_m) > 0$, $l, m = i-1, i$, and $\omega(t) \cdot N_j$, has precisely one sign change in $t \in [t_{i-1}, t_i]$, $m = i-1, i$.*

The above definition requires the projection of curve on the plane perpendicular to N_{i-1} and N_i have only one inflection point. Thus it takes care about inflection preservation along two viewpoints only.

In [7, Goodman and Ong, 1997], [13, Kong and Ong, 2002] we have the following definition

Definition 6.2 *[7, Goodman and Ong, 1997] Inflection preservation criteria is defined by the condition that if $N_{i-1} \cdot N_i < 0$,*

1. *$\omega(t_{i-1}) \cdot N_{i-1} > 0$, $\omega(t_i) \cdot N_i > 0$ and*
2. *for all $N = \lambda N_{i-1} + \mu N_i$, where $\lambda\mu \leq 0$, $(\lambda, \mu) \neq (0, 0)$, $\omega(t) \cdot N$ has precisely one sign change in $[t_{i-1}, t_i]$.*

One can see that the above definition takes care about inflection preservation along all the viewpoints between N_{i-1} and N_i along the plane containing the two normals.

We now observe that our lemma 3.13 acts as the connection between the analysis of inflection of curves and polygonal arcs and the definition 6.2. The two conditions basically states that the projection of the curve $\gamma(t)$ on the plane with normal vector N, $P_{N^\perp}(\gamma(t))$, $t \in [t_{i-1}, t_i]$ should have only one inflection point.

We now state the condition under which a curve segment of $\gamma(t)$ with Bézier representation satisfies the inflection criteria. Let a Bézier curve $\gamma(t)$ be

$$\gamma(t) \quad = \quad \sum_{i=1}^{n} A_i \binom{n}{i} t^i (1-t)^{n-i}, \; 0 \leq t \leq 1, \tag{6.1}$$

$A_i \in R^3$, the polygonal arc $\Gamma = A_0 A_1 \cdots A_n$, $V_i = (A_i - A_{i-1}) \times (A_{i+1} - A_i)$.

If A_i, $i = 0, ..., n$ are such that

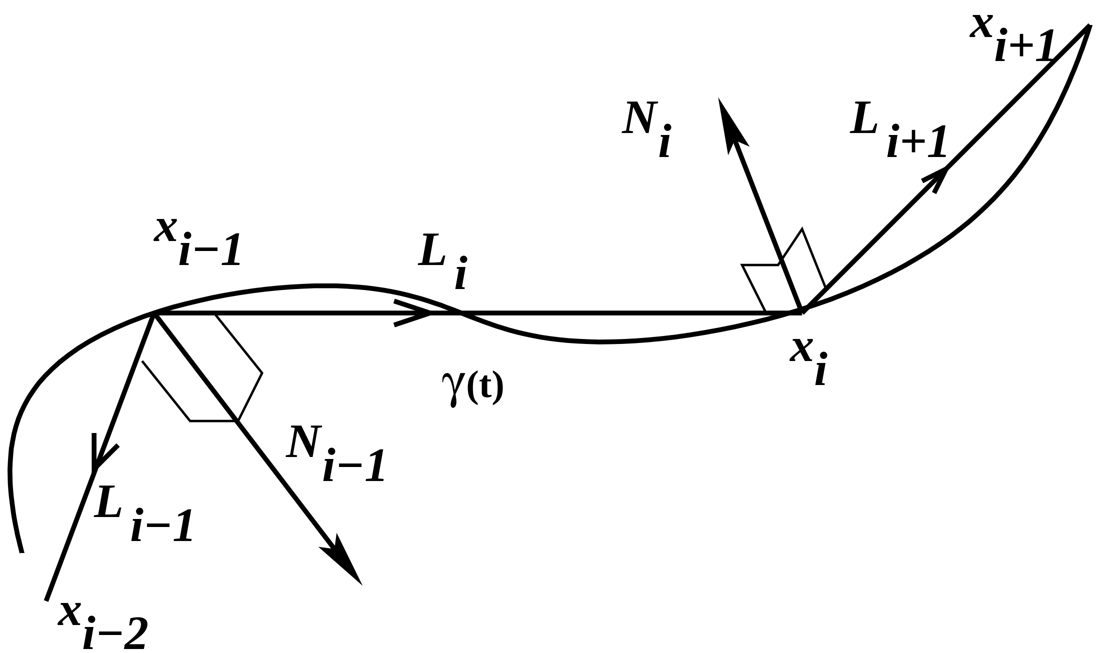


Figure 12: Data point with $N_{i-1} \cdot N_i < 0$ requiring inflection preservation by i^{th} curve segment

1. $N_1 \cdot V_1 < 0$, $N_1 \cdot V_{n-1} > 0$ and $P_{N_1^\perp}$ have only one inflection

2. $N_2 \cdot V_1 > 0$, $N_2 \cdot V_{n-1} < 0$ and $P_{N_1^\perp}$ have only one inflection,

3. $(N_1 \cdot V_i)(N_2 \cdot V_i) < 0, i = 1, ..., n-1$.

and we have two scalars λ, μ such that $\lambda\mu \leq 0$, $(\lambda, \mu) \neq (0, 0)$, $N = \lambda N_i + \mu N_{i+1}$ then we have $V_i \cdot N = \lambda V_i \cdot N_i + \mu V_i \cdot N_{i+1}$ and thus $P_{N^\perp}\Gamma$ has only one inflection point.

Thus we see that if the i^{th} curve segment of interpolating spline $\gamma_i(t)$ is a cubic curve and satisfies the first condition of the definition (6.2) then it also satisfies the second condition.

7 Difficulties in the construction of convexity and inflection preserving splines

We observe that results on inflection counts, apart from affecting the analysis of inflection criteria of splines, have significant effect on the analysis for convexity preservation criteria of splines. Convexity preservation criteria requires that under certain conditions projection of a curve segment $\gamma(t)$ on planes with a specified normal N_c should be convex. Also the condition that inflection count of curve $\gamma(t)$ is greater than 1, that is, $I(\gamma) \geq 1$ says that there exist a vector N_s, such that, projection of curve on planes with normal vector N_s has inflection points

greater than 1 and hence is not convex. Thus we see that if $N_c = N_s$, then $\gamma(t)$ fails to satisfy the convexity criteria.

Among the results stated below some of them are stated in [5, Goodman, 1991] as corollaries we state them as theorems because of their relevance to us.

Theorem 7.1 *[5, Goodman, 1991] If γ is a curve, as in theorem (5.2), which is not planar, then $I(\gamma) \geq 1$.*

Using the theorem (4.6) we have following results for cubic Bézier curves.

Theorem 7.2 *[5, Goodman, 1991] If γ is a cubic polynomial curve which is not planar, $I(\gamma) = 2$*

We have seen in previous sections that convexity and inflection counts Bézier and B-spline curve are related to the convexity and inflection counts of their control polygons. We state few results from [5, Goodman, 1991] for inflection count of polygonal arcs.

Theorem 7.3 *[5, Goodman, 1991] If $P_0, ..., P_n$ are not coplanar, then $I(\Gamma) = 1$ if and only if $V_1, ..., V_{n-1}$ lie in order in a plane sector sub-tending an angle $\leq \pi$.*

Theorem 7.4 *[5, Goodman, 1991] If $n = 3$ and $P_0, ..., P_3$ are not coplanar, then $I(\Gamma) = 1$.*

Remark 7.5 *The above theorems negates the general perception about the inflection counts and convexity of curves and polygonal arcs.*

Remark 7.6 *The proofs provided in [5, Goodman, 1991] for the theorems stated above are constructive, that is, plane on which projection of the curves have inflections points are explicitly constructed.*

We now illustrate the difficulties in constructing a convexity and inflection preserving curve (as indicated in the theorems above) in the following examples.

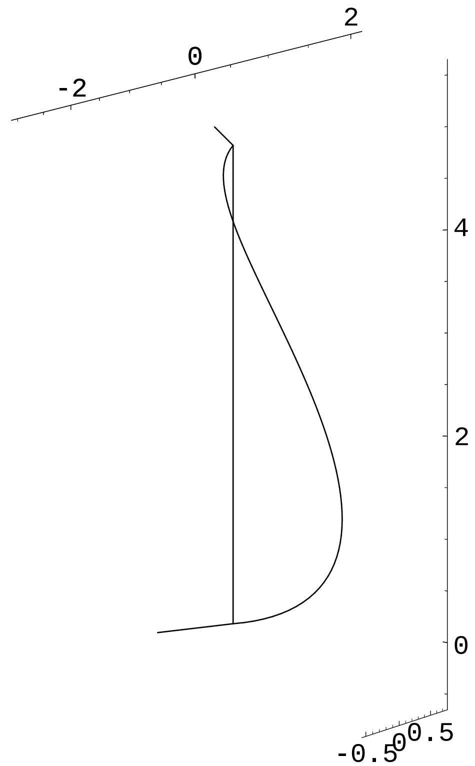
Example 7.7 *Let $\mathbf{x}_{i-2} = (-3, -3, -0.5)$, $\mathbf{x}_{i-1} = (0, 0, 0)$, $\mathbf{x}_i = (0, 0, 5)$, $\mathbf{x}_{i+1} = (2, -4, 5.5)$. The normals at $\mathbf{x}_{i-1}$ and $\mathbf{x}_i$ are $\mathbf{N}_{i-1} = (1.5, -1.5, 0)$ and $\mathbf{N}_i = (2., 1., 0)$ respectively. Since $N_{i-1} \cdot N_i = 1.5 > 0$ we require that the curve between $\mathbf{x}_{i-1}$ and $\mathbf{x}_i$ satisfy convexity preservation criteria.*

Data polygonal arc D formed by $\mathbf{x}_{i-2}$, $\mathbf{x}_{i-1}$, $\mathbf{x}_i$ and $\mathbf{x}_{i+1}$ along with normals $\mathbf{N}_{i-1}$ and $\mathbf{N}_i$ as thick lines is shown in figure 13(a). The figures 13(b), 13(c) and 13(d) show the curve between $\mathbf{x}_{i-1}$ and $\mathbf{x}_i$ with data polygonal arc D along the viewpoints $\mathbf{N}_{i-1}$ and $\mathbf{N}_i$ and $V_1 = (-10, 0, 1)$ (that is projection in the plane with normals $\mathbf{N}_{i-1}$, $\mathbf{N}_i$ and $\mathbf{V}_1$). We observe that though the curve satisfies the conditions of convexity preservation criteria along the viewpoint $\mathbf{V}_1$, it fails to do the same for both the viewpoints $\mathbf{N}_{i-1}$ and $\mathbf{N}_i$. Thus we see that it is relatively difficult to construct a curve satisfying the convexity preservation criteria using a graphical interface.

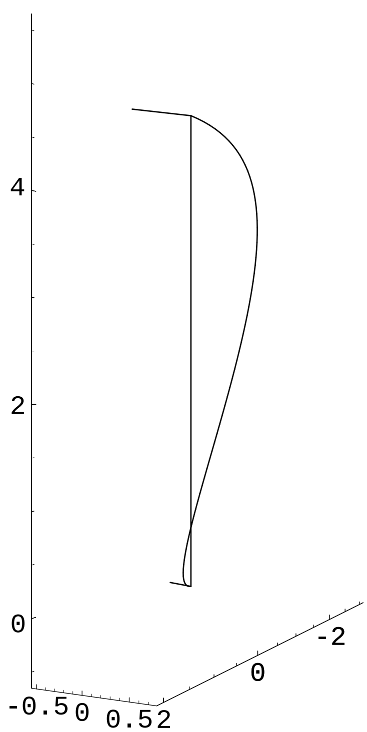
The mathematica code for the generation of figures 13(a), 13(b), 13(c) and 13(c) is as below:

```
(***********************************************************)
m1={3,1,0.5};m2={2,-1,0.5};
A={0,0,0}; B={0,0,5}; Am=A+m1; Bm=B-m2;
P=((1-t)^3)*A + (3*(1-t)^2*t)*Am + (3*(1-t)*t^2)*Bm + (t^3)*B
L0={-3,-3,-0.5}; L1=B-A; L2={2,-4,0.5}; F=B+L2;
N1=(1/10)*Cross[(-1)*L0,B] N2=(1/10)*Cross[B,L2] DT=N1.N2
P0=A+t*L0; P1=A+t*B; P2=B+t*L2;
(* L0, A, B,F are data points *)
Poly=Show[Graphics3D[{Line[{L0,A,B,F}], {Thickness[0.010], Line[{A,N1}]}, {Thickness[0.010], Line[{B,N2}]}}],
ViewPoint->{-1,0,1}, Boxed->False];
Poly0=Show[Graphics3D[Line[{L0,A,B,F}]], ViewPoint->{-10,0,1}, Boxed->False];
ln0=ParametricPlot3D[P0,{t,0,1}, Boxed->False];
ln1=ParametricPlot3D[P1,{t,0,1}, Boxed->False];
ln2=ParametricPlot3D[P2,{t,0,1}, Boxed->False];
Crv0=ParametricPlot3D[P,{t,0,1}, ViewPoint->N1, Boxed->False];
CrvPoly0=Show[{Crv1,Poly0}, ViewPoint->N1, Boxed->False];
Crv1=ParametricPlot3D[P, {t,0,1}, ViewPoint->N2, Boxed->False];
CrvPoly1=Show[{Crv1,Poly0}, ViewPoint->N2, Boxed->False];
Crv2=ParametricPlot3D[P,{t,0,1}, ViewPoint->{-10,0,1}, Boxed->False];
CrvPoly2=Show[ln0,ln1,ln2,Crv2];
Display["d:\Gautam_Viewpoint\view0.png", Poly, "PNG"];
Display["d:\Gautam_Viewpoint\view1.png", CrvPoly0, "PNG"];
Display["d:\Gautam_Viewpoint\view2.png", CrvPoly1, "PNG"];
```

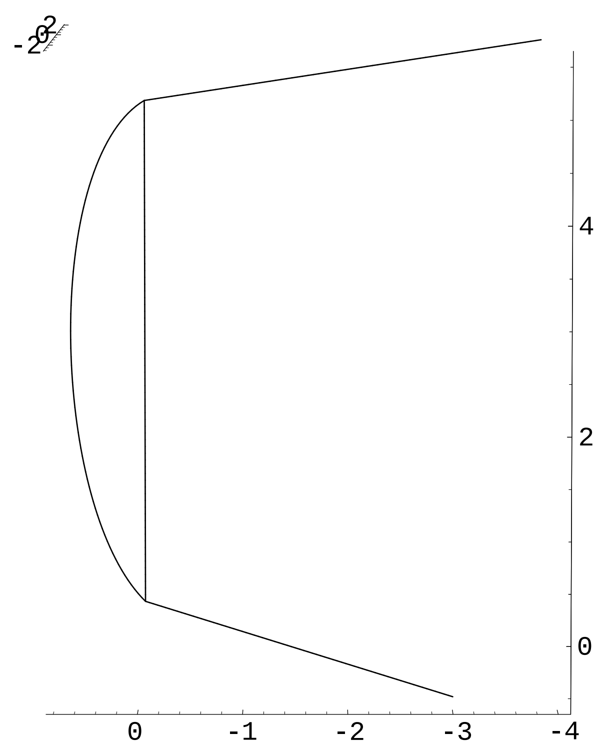
(a) Data polygon arc with normals as thick lines



(b) Viewpoint $\mathbf{N}_{i-1}$



(c) Viewpoint $\mathbf{N}_{i-1}$



(d) Viewpoint $\mathbf{V}_1 = (-10, 0, 1)$

Figure 13: Data polygonal arc with curve between $\mathbf{x}_{i-1}$ and $\mathbf{x}_i$ along different viewpoints

Display["d:\Gautam_Viewpoint\view3.png", CrvPoly2, "PNG"];

(***)

Example 7.8 *Here we have* $\mathbf{x}_{i-2} = (-3,-3,-0.5)$, $\mathbf{x}_{i-1} = (0,0,0)$, $\mathbf{x}_i = (0,0,10)$, $\mathbf{x}_{i+1} = (2,-4,10.5)$. *The normals at* $\mathbf{x}_{i-1}$ *and* $\mathbf{x}_i$ *are* $\mathbf{N}_{i-1} = (30.,-30.,0)$ *and* $\mathbf{N}_i = (40.,20.,0)$ *respectively. Since* $N_{i-1} \cdot N_i = 600 > 0$ *we require that the curve between* $\mathbf{x}_{i-1}$ *and* $\mathbf{x}_i$ *satisfy convexity preservation criteria.*

The figures 14(a), 14(b), 14(c), 14(d) 14(e) and 14(f) show the curve between $\mathbf{x}_{i-1}$ *and* $\mathbf{x}_i$ *with data polygonal arc along the viewpoints* $\mathbf{N}_{i-1}$, $\mathbf{N}_i$, $\mathbf{V}_1 = (7.458,-1.863,-3.506)$, $\mathbf{V}_2 = (-17.458,1.863,23.506)$, $\mathbf{V}_3 = (-7.458,6.863,33.506)$ *and* $\mathbf{V}_4 = (-7.458,30.863,33.506)$ *respectively. Observe that the curve along viewpoints* $\mathbf{N}_{i-1}$, $\mathbf{N}_i$, $\mathbf{V}_1$, $\mathbf{V}_2$ *is convex where as along viewpoints* $\mathbf{V}_3$ *and* $\mathbf{V}_4$ *is not convex. This means that if values of* $\mathbf{x}_{i-2}$ *or* $\mathbf{x}_{i+1}$ *are altered such that* $\mathbf{N}_{i-1}$ *or* $\mathbf{N}_i$ *changes to* $\mathbf{V}_3$ *or* $\mathbf{V}_4$ *with* $\mathbf{N}_{i-1} \cdot \mathbf{N}_i > 0$*, then the curve doesn't satisfy convexity preservation criteria with respect to changed data polygonal arc.*

In addition to the above observation in example we also note that (figure 14(e)) along the viewpoint $\mathbf{V}_3 = (-7.458,6.863,33.506)$ *the curve has two inflections as shown in figure 15 (the curvature of the projection of curve along the viewpoint* $\mathbf{V}_5$ *changes its sign twice). Thus if* $\mathbf{x}_{i-2}$ *or* $\mathbf{x}_{i+1}$ *are such that* $\mathbf{N}_{i-1}$ *or* $\mathbf{N}_i$ *respectively are equal to* $\mathbf{V}_3$*, with* $\mathbf{N}_{i-1} \cdot \mathbf{N}_i < 0$ *then the curve would not satisfy inflection preservation criteria.*

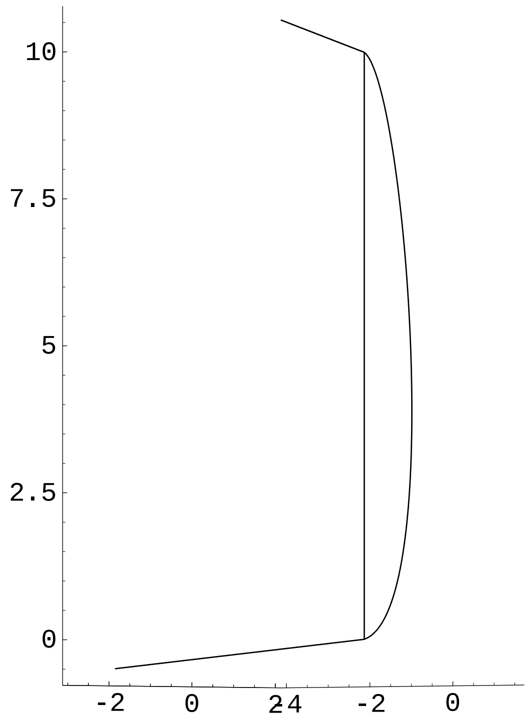
The mathematica code for the generation of figures 14(a), 14(b), 14(c), 14(d), 14(e), 14(f) is as below:

```
(****************************************************)
m1x=1;m1y=1;m1z=0.5;
m2x=2;m2y=-3;m2z=0.5;
Ax=0;Ay=0;Az=0;
Dx=0;Dy=0;Dz=10;
Bx=Ax+m1x; By=Ay+m1y;Bz=Az+m1z;(*(1,1,0.5)*)
Cx=Dx-m2x;Cy=Dy-m2y; Cz=Dz-m2z;(*(-2,3,9.5)*)
Px=Ax*(1-t)^ 3 + Bx*3*(1-t)^ 2*t + Cx*3*(1-t)*t^ 2 + Dx*t^ 3;
Py=Ay*(1-t)^ 3 + By*3*(1-t)^ 2*t + Cy*3*(1-t)*t^ 2 + Dy*t^ 3;
Pz=Az*(1-t)^ 3 + Bz*3*(1-t)^ 2*t + Cz*3*(1-t)*t^ 2 + Dz*t^ 3;
L0x=-3;L0y=-3;L0z=-0.5;(*xi-2=(-3,-3,-0.5)*)
L1x=Dx-Ax;L1y=Dy-Ay;L1z=Dz-Az;
```

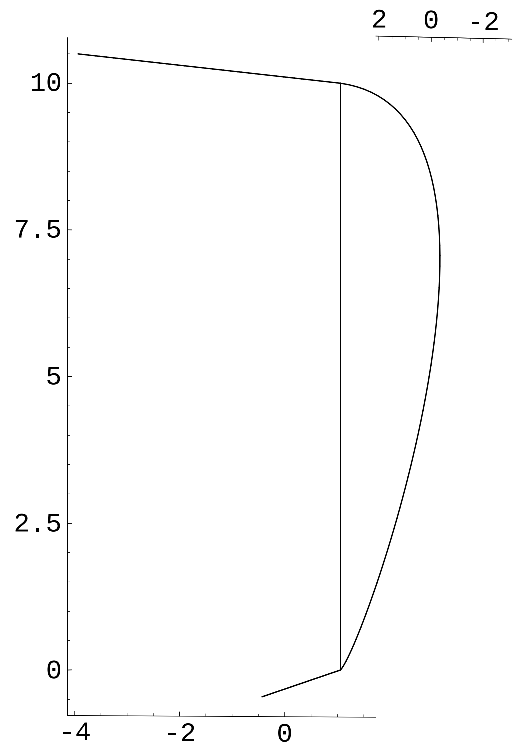
```
L2x=2;L2y=-4;L2z=0.5;
P0x=Ax+L0x*t;P0y=Ay+L0y*t;P0z=Az+L0z*t;
P1x=Ax+Dx*t;P1y=Ay+Dy*t;P1z=Az+Dz*t;
P2x=Dx+L2x*t;P2y=Dy+L2y*t;P2z=Dz+L2z*t;(*xi+1=(2,-4,10.5)*)
N1=Cross[{-L0x,-L0y,-L0z},{0,0,10}] N2=Cross[{0,0,10},{L2x,L2y,10+L2z}] A=N1.N2
CrvPoly1=ParametricPlot3D[{{Px,Py,Pz}, {P0x,P0y,P0z}, {P1x,P1y,P1z}, {P2x,P2y,P2z}}, {t,0,1}, ViewPoint-
Boxed- >False];
CrvPoly2=ParametricPlot3D[{{Px,Py,Pz}, {P0x,P0y,P0z}, {P1x,P1y,P1z}, {P2x,P2y,P2z}}, {t,0,1}, ViewPoint-
Boxed- >False];
CrvPoly3=ParametricPlot3D[{{Px,Py,Pz}, {P0x,P0y,P0z}, {P1x,P1y,P1z}, {P2x,P2y,P2z}}, {t,0,1}, ViewPoint-
-1.863, -3.506}, Boxed- >False];
CrvPoly4=ParametricPlot3D[{{Px,Py,Pz}, {P0x,P0y,P0z}, {P1x,P1y,P1z}, {P2x,P2y,P2z}}, {t,0,1}, ViewPoint-
17.458, 1.863, 23.506}, Boxed- >False];
CrvPoly5=ParametricPlot3D[{{Px,Py,Pz}, {P0x,P0y,P0z}, {P1x,P1y,P1z}, {P2x,P2y,P2z}}, {t,0,1}, ViewPoint-
7.458, 6.863, 33.506}, Boxed- >False];
CrvPoly6=ParametricPlot3D[{{Px,Py,Pz}, {P0x,P0y,P0z}, {P1x,P1y,P1z}, {P2x,P2y,P2z}}, {t,0,1}, ViewPoint-
7.458, 30.863, 33.506}, Boxed- >False];
Display["newview1.png",CrvPoly1,"PNG"]
Display["newview2.png",CrvPoly2,"PNG"]
Display["newview3.png",CrvPoly3,"PNG"]
Display["newview4.png",CrvPoly4,"PNG"]
Display["newview5.png",CrvPoly5,"PNG"]
Display["newview6.png",CrvPoly6,"PNG"]
(*****************************************************)
```

The mathematica code for the generation of figure 15 for the curvature of the projection of the curve along the viewpoint $\mathbf{V}_3$ *is as below:*

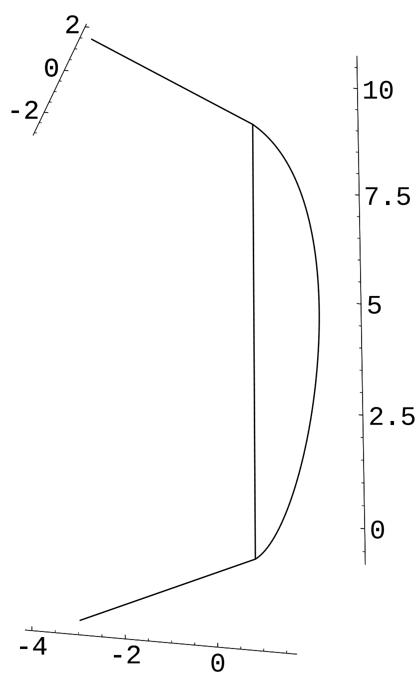
```
(*********************************************************)
m1={1,1,0.5}; m2={2,-3,0.5};
A={0,0,0}; B={0,0,10}; Am=A+m1; Bm=B-m2;
P=((1-t)^ 3)*A + (3*(1-t)^ 2*t)*Am + (3*(1-t)*t^ 2)*Bm + (t^ 3)*B
DR1=D[P,{t,1}] DR2=D[P,{t,2}] CTR=Cross[DR1,DR2]
CTRPerp=CTR.{-7.458, 6.863, 33.506}
```



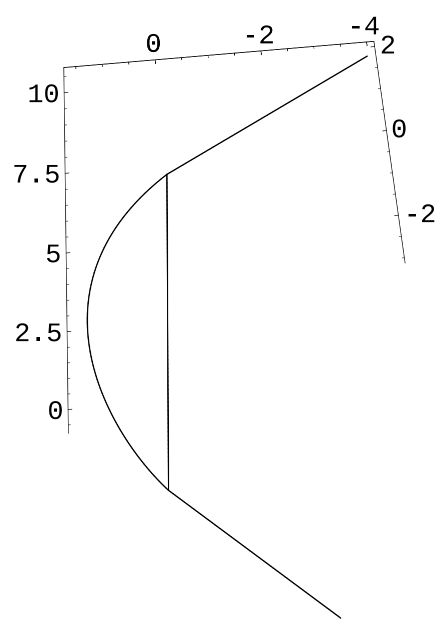
(a) Viewpoint $\mathbf{N}_{i-1}$



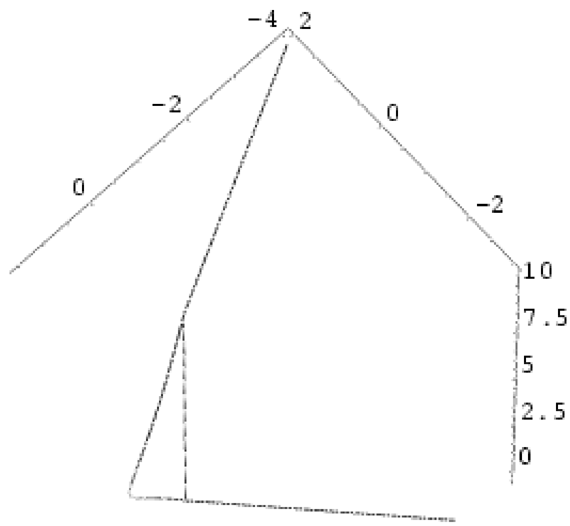
(b) Viewpoint $\mathbf{N}_i$



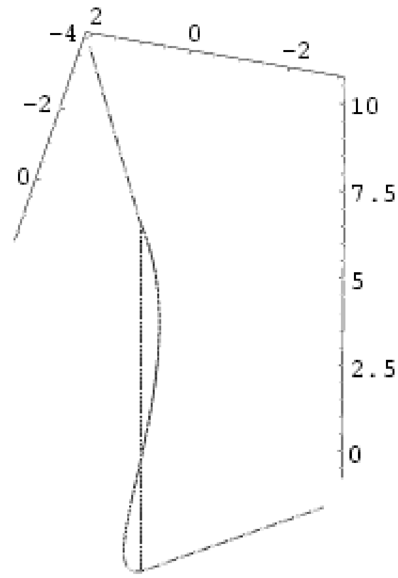
(c) Viewpoint $\mathbf{V}_1 = (7.458, -1.863, -3.506)$



(d) $\mathbf{V}_2 = (-17.458, 1.863, 23.506)$



(e) Viewpoint $\mathbf{V}_3 = (-7.458, 6.863, 33.506)$



(f) Viewpoint $\mathbf{V}_4 = (-7.458, 30.863, 33.506)$

Figure 14: Data polygonal arc with curve between $\mathbf{x}_{i-1}$ and $\mathbf{x}_i$ along different viewpoints

Infl5=Plot[CTRPerp,{t,0,1}]

Display["d:\Gautam_Viewpoint\infl5.png", Infl5, "PNG"];

(***)

Remark 7.9 *Note in that in the above examples as we change the viewpoints the curve and the data polygonal arc approximately same nature of deviation from convexity. In figures the curve as well as data polygonal arc shows approximately same inflections. This due to the fact that the curve satisfies torsion preservation criteria described in section along with convexity preservation criteria.*

8 Collinearity preservation criteria for interpolating splines

Definition 8.1 *[9, Karavelas and Kaklis, 2000] The collinearity preservation criteria is defined by the condition that if $|N_i| = 0$ and $L_{i-1} \cdot L_i > 0$, then*

$$\frac{|\gamma'(t) \times L_j|}{|\gamma'(t)||L_j|} < \epsilon_0, t \in \eta_i, j = i-1, i, \tag{8.1}$$

where ϵ_0 is a user-specified small positive number in $(0, 1]$, and η_i a user specified closed subinterval of (t_{i-1}, t_{i+1}) that includes t_i as an interior point.

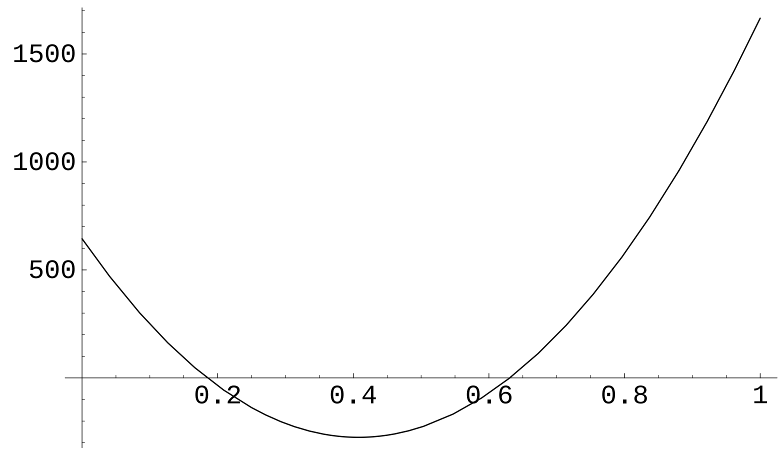


Figure 15: Curvature of the projection of the curve along the viewpoint $\mathbf{V}_3$

We note that $|N_i| = 0 \implies L_{i-1} = \alpha L_i$, $\alpha \in R$ and this condition with $L_{i-1} \cdot L_i > 0$ implies that $L_{i-1} = \alpha L_i$, $\alpha \in R^+$. Equation (8.1) states that the (sine of the) angle between tangent vector at each point on the spline $\gamma(t)$ and L is less than $\epsilon > 0$ in the user specified closed interval in (t_{i-1}, t_{i+1}). Thus collinearity preservation criteria requires that if two consecutive polygon segments are collinear and is having the same direction then the curve segments of the corresponding indexes should be approximately collinear and parallel to the corresponding polygon segments.

We now investigate collinearity preservation criteria a bit more closely. One natural question to ask is: What if one considers $\mathbf{x}_i$ to be redundant? Well in that case, if the curve $\gamma(t)$ is not collinear to the line segment $\{\mathbf{x}_{i-1}, \mathbf{x}_{i+1}\}$, then the curve between $\mathbf{x}_{i-1}$ and $\mathbf{x}_{i+1}$ is required to satisfy other shape preservation criteria. For convenience of understanding the situation let us suppose $\mathbf{x}_{i-2}$, $\mathbf{x}_{i-1}$, $\mathbf{x}_i$, $\mathbf{x}_{i+1}$, $\mathbf{x}_{i+2}$ are coplanar. Now consider the following cases

case i $N_{i-1} \cdot N_{i+1} \geq 0$ to be refered as convex neighbourhood data

case ii $N_{i-1} \cdot N_{i+1} < 0$ to be refrered as inflection neighbourhood data

For **case i** we propose that if the curve does not coincide with line segments $\{\mathbf{x}_{i-1}, \mathbf{x}_i\}$, $\{\mathbf{x}_i, \mathbf{x}_{i+1}\}$ one must ensure that

1. $\gamma(t)$ does not interpolate $\mathbf{x}_i$,

2. $\omega(t_j) \cdot N_j \geq 0$, $j = i-1, i+1$,

3. $(\gamma'(t_{i+1}) \times L_{i+1}) \cdot (\gamma'(t_{i+1}) \times L_{i+2}) < 0$ and $(\gamma'(t_{i-1}) \times L_i) \cdot (\gamma'(t_{i-1}) \times L_{i-1}) < 0$

4. $\gamma(t)$ is globally convex between $\mathbf{x}_{i-1}$ and $\mathbf{x}_{i+1}$,

5. $\gamma'(t) = \alpha L_i$, $\alpha \in R^+$ for $t \in (t_i - \eta, t_i + \delta) \subset [t_{i-1}, t_{i+1}]$ (suitable choice of η and δ provides necessary tilt to the curve $\gamma(t)$).

For a suitable choice of ϵ_0, condition 8.1 along with conditions 1-5 the curve $\gamma(t)$ have following properties (see Figure 1 and 2):

- satisfies convexity preservation criteria between $\mathbf{x}_{i-1}$ and $\mathbf{x}_{i+1}$ and

- convexity and inflection criteria preservation criteria for data arc segments $\{\mathbf{x}_{i-1}, \mathbf{x}_{i-2}\}$ and $\{\mathbf{x}_{i+1}, \mathbf{x}_{i+2}\}$ achievable.

Violation of any of these conditions leads to the violation of the above properties as illustrated by Figure 3 and collinconvwr2.

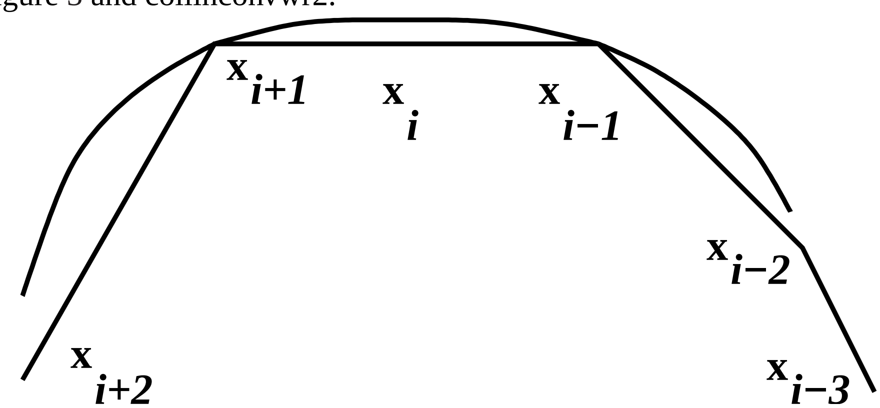


Figure 16a : Collinearity preservation criteria for convex neighbourhood data makes conditions for convexity preservation criteria between $\mathbf{x}_{i-1}$ and $\mathbf{x}_{i-2}$ achievable.

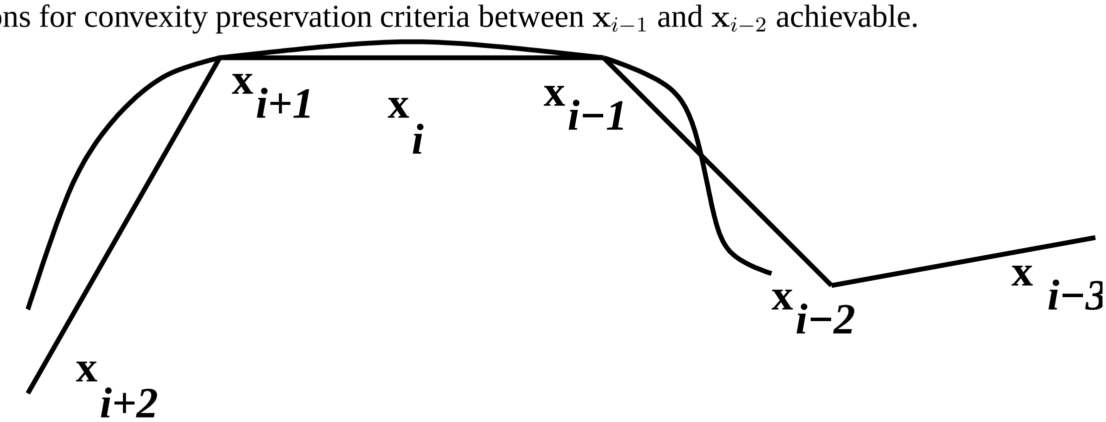


Figure 16b : Collinearity preservation criteria for convex neighbourhood data makes conditions for inflection preservation criteria between $\mathbf{x}_{i-1}$ and $\mathbf{x}_{i-2}$ achievable.

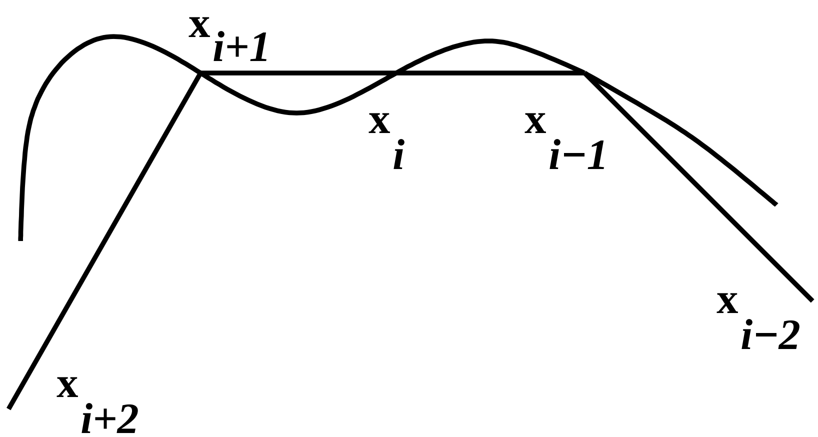


Figure 16c : Violation of collinearity preservation criteria for convex neighbourhood data leads to violation of convexity preservation criteria between $\mathbf{x}_{i-1}$ and $\mathbf{x}_{i+1}$

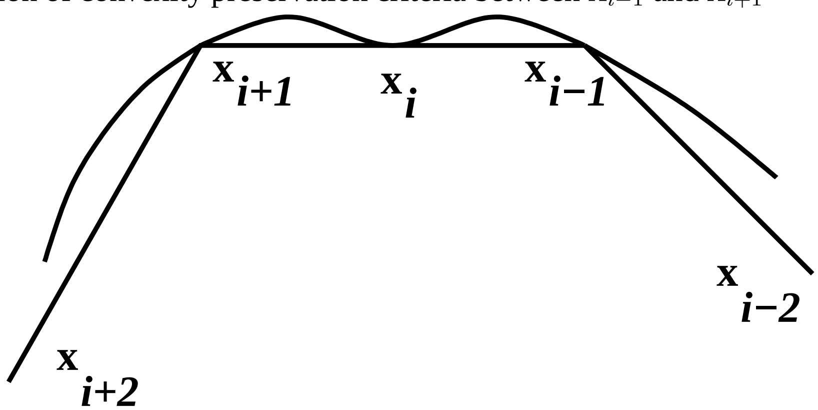


Figure 16d : Violation of collinearity preservation criteria for convex neighbourhood data leads to violation of convexity preservation criteria between $\mathbf{x}_{i-1}$ and $\mathbf{x}_{i+1}$

For **case ii** we propose that if the curve does not coincide with line segments $\{\mathbf{x}_{i-1}, \mathbf{x}_i\}$, $\{\mathbf{x}_i, \mathbf{x}_{i+1}\}$ one must ensure that

1. $\gamma(t)$ interpolate $\mathbf{x}_i$,
2. $\omega(t_j) \cdot N_j \geq 0$, $j = i-1, i+1$,
3. $(\gamma'(t_{i+1}) \times L_{i+1}) \cdot (\gamma'(t_{i+1}) \times L_{i+2}) < 0$ and $(\gamma'(t_{i-1}) \times L_i) \cdot (\gamma'(t_{i-1}) \times L_{i-1}) < 0$
4. $\omega(t)$ changes sign only once between $\mathbf{x}_{i-1}$ and $\mathbf{x}_{i+1}$
5. $\omega(t)$ changes sign at $t = t_i$.

For a suitable choice of ϵ_0, condition 8.1 along with conditions 1-5 the curve $\gamma(t)$ have following properties (see Figure 1 and 2):

- satisfies convexity preservation criteria between $\mathbf{x}_{i-1}$ and $\mathbf{x}_{i+1}$ and

- convexity and inflection criteria preservation criteria for data arc segments $\{\mathbf{x}_{i-1}, \mathbf{x}_{i-2}\}$ and $\{\mathbf{x}_{i+1}, \mathbf{x}_{i+2}\}$ achievable.

Violation of any of these conditions leads to the violation of the above properties as illustrated by 3 and 4 .

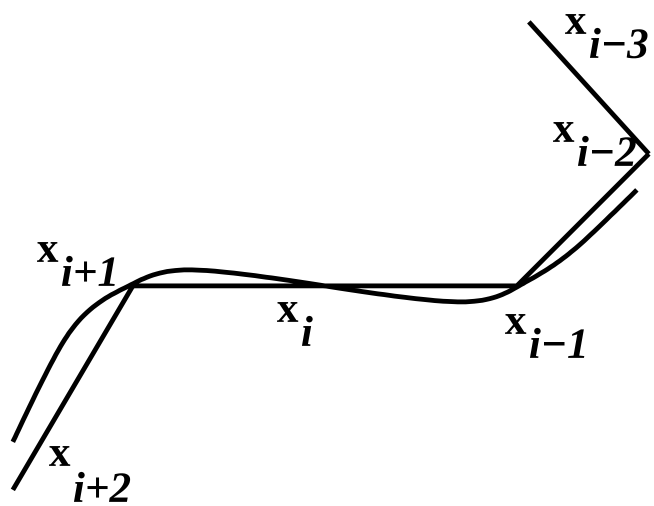


Figure 17a : Collinearity preservation criteria for inflection neighbourhood data makes conditions for convexity preservation criteria between $\mathbf{x}_{i-1}$ and $\mathbf{x}_{i-2}$ achievable.

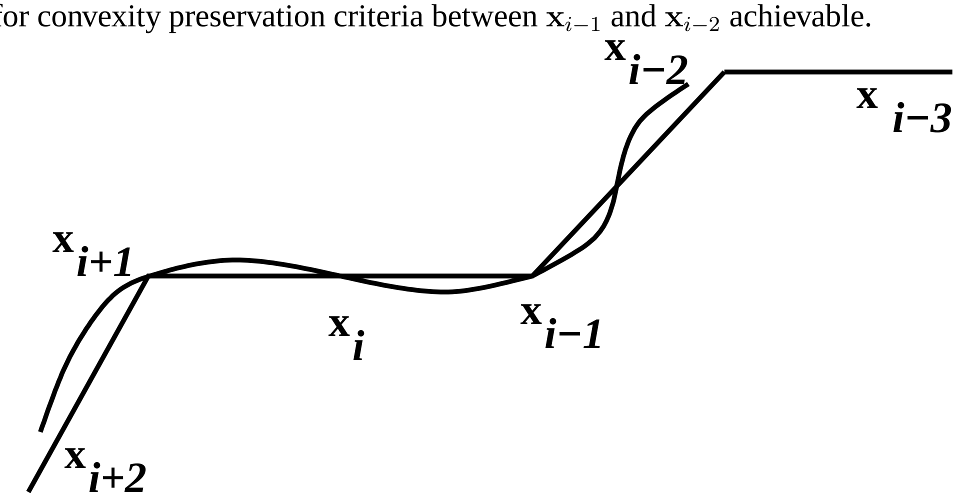


Figure 17b : Collinearity preservation criteria for inflection neighbourhood data makes conditions for inflection preservation criteria between $\mathbf{x}_{i-1}$ and $\mathbf{x}_{i-2}$ achievable.

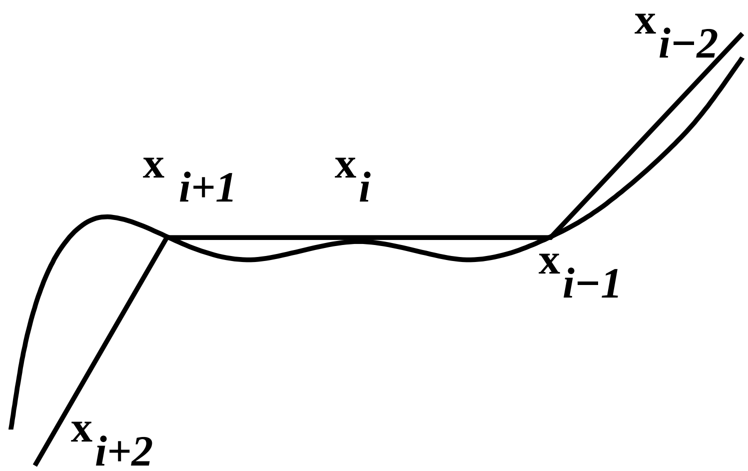


Figure 17c : Violation of collinearity preservation criteria for inflection neighbourhood data leads to violation of inflection preservation criteria between $\mathbf{x}_{i-1}$ and $\mathbf{x}_{i+1}$

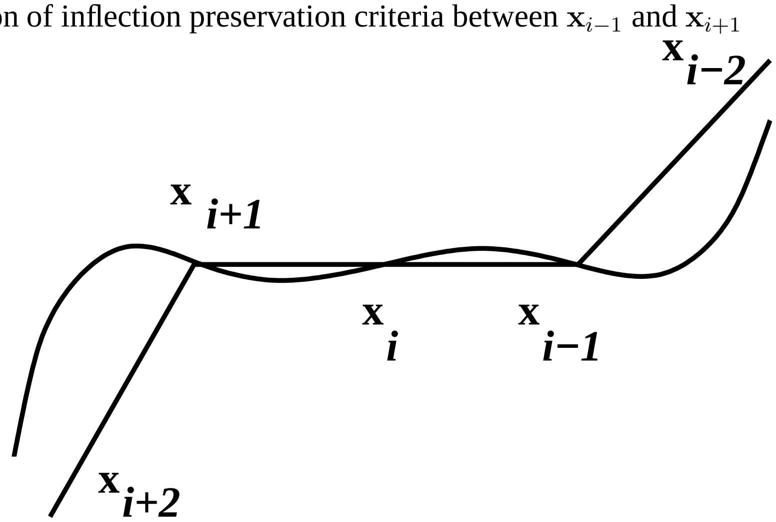


Figure 17d : Violation of collinearity preservation criteria for inflection neighbourhood data leads to violation of inflection preservation criteria between $\mathbf{x}_{i-1}$ and $\mathbf{x}_{i+1}$

Thus we see that we need to modify the definition of collinearity preservation criteria for the general data, that is, when $\mathbf{x}_{i-2}$ $\mathbf{x}_{i-1}$, $\mathbf{x}_i$, $\mathbf{x}_{i+1}$ and $\mathbf{x}_{i+2}$ are nonplanar. The modification is to be done by adding conditions according to the following cases:

case i $N_{i-1} \cdot N_{i+1} \geq 0$ to be refered as convex neighbourhood data

case ii $N_{i-1} \cdot N_{i+1} < 0$ to be refrered as inflection neighbourhood data

We state our modified definition as follows:

Definition 8.2 *The collinearity preservation criteria is defined by the condition that if* $|N_i| = 0$ *and* $L_{i-1} \cdot L_i > 0$*, then*

$$\frac{|\gamma'(t) \times L_j|}{|\gamma'(t)||L_j|} < \epsilon_0, t \in \eta_i, j = i-1, i, \tag{8.2}$$

where ϵ_0 is a user-specified small positive number in $(0, 1]$, and η_i a user specified closed subinterval of (t_{i-1}, t_{i+1}) that includes t_i as an interior point and additionally for the case of convex neighbourhood data

1. *$\gamma(t)$ does not interpolate $\mathbf{x}_i$,*

2. *$\gamma(t)$ should satisfy convexity preservation criteria between $\mathbf{x}_{i-1}$ and $\mathbf{x}_{i+1}$, considering $\mathbf{x}_{i-2}$, $\mathbf{x}_{i-1}$, $\mathbf{x}_{i+1}$ and $\mathbf{x}_{i+2}$ as consecutive data points,*

3. *$|\gamma'(t_{i+1}) \times L_{i+1}||\gamma'(t_{i+1}) \times L_{i+2}| < 0$ and $|\gamma'(t_{i-1}) \times L_i||\gamma'(t_{i-1}) \times L_{i-1}| < 0$*

4. *$\gamma'(t) = \alpha L_i$, $\alpha \in R^+$ for $t \in (t_i - \eta, t_i + \delta) \subset [t_{i-1}, t_{i+1}]$ (suitable choice of η and δ provides necessary tilt to the curve $\gamma(t)$).*

for the case of inflection neighbourhood data

1. *$\gamma(t)$ interpolate $\mathbf{x}_i$,*

2. *$\gamma(t)$ should satisfy inflection preservation criteria between $\mathbf{x}_{i-1}$ and $\mathbf{x}_{i+1}$, considering $\mathbf{x}_{i-2}$, $\mathbf{x}_{i-1}$, $\mathbf{x}_{i+1}$ and $\mathbf{x}_{i+2}$ as consecutive data points,*

3. *$|\gamma'(t_{i+1}) \times L_{i+1}||\gamma'(t_{i+1}) \times L_{i+2}| < 0$ and $|\gamma'(t_{i-1}) \times L_i||\gamma'(t_{i-1}) \times L_{i-1}| < 0$*

Considering $\mathbf{x}_{i-2}$, $\mathbf{x}_{i-1}$, $\mathbf{x}_{i+1}$ and $\mathbf{x}_{i+2}$ as consecutive data points the curve $\gamma(t)$ between $\mathbf{x}_{i-1}$ and $\mathbf{x}_{i+1}$ also satisfy torsion preservation criteria (to be stated in section 9) or coplanarity preservation criteria (to be stated in section 10) according to the condition $[L_{i-1}\ L_i\ L_{i+1}] \neq 0$ or $[L_{i-1}\ L_i\ L_{i+1}] = 0$ respectively.

We observe that the last two conditions guides the spatial behavior of the curve $\gamma(t)$ between $\mathbf{x}_{i-1}$ and $\mathbf{x}_{i+1}$, with respect to the planes Π_{i-1} (containing $\mathbf{x}_{i-2}$, $\mathbf{x}_{i-1}$, and $\mathbf{x}_i$) and Π_{i+1} (containing $\mathbf{x}_i$, $\mathbf{x}_{i+1}$ and $\mathbf{x}_{i+2}$). We also observe that in case $\mathbf{x}_{i-2}$, $\mathbf{x}_{i-1}$, $\mathbf{x}_i$, $\mathbf{x}_{i+1}$ and $\mathbf{x}_{i+2}$ are nonplanar, then conditions of the definition 8.2 makes conditions of coplanarity preservation criteria (to be stated in section 10) for curve $\gamma(t)$ between $\mathbf{x}_{i+2}$ and $\mathbf{x}_{i+3}$ and between $\mathbf{x}_{i-2}$ and $\mathbf{x}_{i-1}$, achievable. Thus we see that for a person, who tends to ignore $\mathbf{x}_i$ as a data point and considers $\mathbf{x}_{i-1}$ and $\mathbf{x}_{i+1}$ as adjacent data points, definition 8.2 makes all the shape preservation criteria by $\gamma(t)$ between $\mathbf{x}_{i-1}$ and $\mathbf{x}_{i+1}$ achievable, without conflict with the shape preservation criteria for curve segment between $\mathbf{x}_{i-2}$ and $\mathbf{x}_{i-1}$ and between $\mathbf{x}_{i+1}$ and $\mathbf{x}_{i+2}$.

9 Torsion preservation criteria for interpolating splines

Definition 9.1 *Discrete torsion for the polygonal arc* $\mathbf{x}_0\mathbf{x}_1\cdots\mathbf{x}_n$ *is defined as*

$$\triangle_i = [L_{i-1}\ L_i\ L_{i+1}],\ i = 3, \ldots, n-1 \tag{9.1}$$

where $L_i = \mathbf{x}_i - \mathbf{x}_{i-1}$, $i = 1, ..., n$

Definition 9.2 *[14, Costantini and Manni, 2003] Torsion preservation criteria consists of following conditions*

1. $\tau_i(t)\triangle_i > 0$ *in a chosen closed subinterval of* (t_{i-1}, t_i), *whenever* $\triangle_i \neq 0$.

2. $\tau_i(t_{i-1})\triangle_j > 0$, $j = i-1, i$, *whenever* $\triangle_{i-1}\triangle_i > 0$

where $\tau_i(t) = \dfrac{|\gamma_i'(t)\ \gamma_i''(t)\ \gamma_i'''(t)|}{\|\gamma_i'(t) \times \gamma_i''(t)\|^2}$, *if* $\gamma_i'(t) \times \gamma_i''(t) \neq 0$.

First condition of torsion preservation criteria states that i^{th} curve segment should appear to twist away from its osculating plane in the same way as L_{i+1} moves away from the plane of $\{\mathbf{x}_{i-2}, \mathbf{x}_{i-1}, \mathbf{x}_i\}$.

Most of the author in their papers do not consider second condition in their definition for torsion preservation criteria. Following definition is followed by them

Definition 9.3 *[13, Kong and Ong, 2002] Torsion preservation criteria is defined by the condition that if* $\triangle_i \neq 0$ *then* $\tau_i(t)\triangle_i \geq 0$, $t \in [t_{i-1}, t_i]$, *where* $\tau_i(t) = \dfrac{|\gamma_i'(t)\ \gamma_i''(t)\ \gamma_i'''(t)|}{\|\gamma_i'(t) \times \gamma_i''(t)\|^2}$, *if* $\gamma_i'(t) \times \gamma_i''(t) \neq 0$.

(We discuss the situation in which $\tau_i(t)\triangle_i = 0$ in the theorem 12.1.)

We now discuss the second condition of definition of 9.2. In figure 18 set of points with bigger circle, $\{\mathbf{x}_{i-2}$, $\mathbf{x}_{i-1}$, $\mathbf{x}_i$, $\mathbf{x}_{i+1}\}$, correspond to $\triangle_i$ and set of points with smaller circle, $\{\mathbf{x}_{i-3}$, $\mathbf{x}_{i-2}$, $\mathbf{x}_{i-1}$, $\mathbf{x}_i\}$, correspond to $\triangle_{i-1}$. We know that the sign of $\triangle_i$ and $\triangle_{i-1}$ depends on the cosine of the angle that L_{i+1} and L_{i-2}, respectively, makes with the normal of the plane Π_{i-1} containing L_{i-1} and L_i. The condition $\triangle_i\triangle_{i-1} > 0$ states that L_{i-2} moves into the plane Π_{i-1} in the same way as L_{i+1} moves out of the plane, that is, the line segments $\{\mathbf{x}_{i-3}, \mathbf{x}_{i-2}\}$ and $\{\mathbf{x}_i$, $\mathbf{x}_{i+1}\}$ lies on the opposite sides of the plane Π_{i-1}. The condition $\tau_i(t_{i-1})\triangle_i > 0$, states that the curve at $t = t_{i-1}$ moves out of its osculating plane in the same way as the vector L_{i+1} moves out of the plane Π_{i-1}. Similarly, the condition $\tau_i(t_{i-1})\triangle_{i-1} > 0$, states that the curve at $t = t_{i-1}$

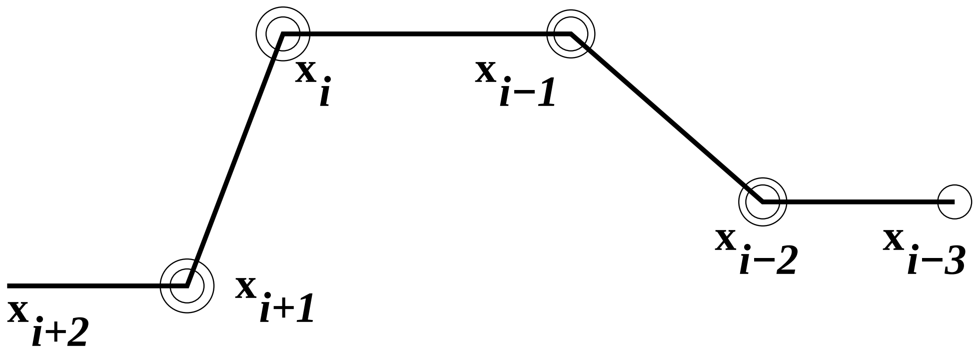


Figure 18: Data polygon arc involved in second condition of definition 9.2

moves out of its osculating plane in the same way as the vector L_{i-2} moves into the plane Π_{i-1} or L_i moves out of the plane Π_{i-2}.

We first observe that sign$(\tau_i(t_{i-1})\triangle_{i-1})$ = sign$(\tau_i(t_{i-1})\triangle_i)$, whenever $\triangle_{i-1}\triangle_i > 0$ and sign$(\tau_i(t_{i-1})\triangle_{i-1}) = -$sign$(\tau_i(t_{i-1})\triangle_i)$, whenever $\triangle_{i-1}\triangle_i < 0$ (since sign$(\tau_i(t_{i-1})\triangle_{i-1})$ = sign$(\tau_i(t_{i-1})\triangle_{i-1}(\triangle_i)^2)$ = sign$(\tau_i(t_{i-1})\triangle_i(\triangle_{i-1}\triangle_i))$).
Given $\triangle_{i-1}\triangle_i > 0$, the condition $\tau_i(t_{i-1})\triangle_{i-1} > 0$ implies that $\tau_i(t_{i-1})\triangle_i > 0$ and vice-versa. The conditions $\tau_i(t_{i-1})\triangle_i > 0$ and $\triangle_{i-1}\triangle_i < 0$ implies that $\tau_i(t_{i-1})\triangle_i < 0$. For the case $\triangle_{i-1}\triangle_i < 0$, that is, the line segments $\{\mathbf{x}_{i-3}, \mathbf{x}_{i-2}\}$ and $\{\mathbf{x}_i, \mathbf{x}_{i+1}\}$ lying on the same side of the plane Π_{i-1}, the curve satisfying the condition $\tau_i(t_{i-1})\triangle_i > 0$, have the property that

• (since condition $\tau_i(t_{i-1})\triangle_i > 0$) the curve at $t = t_{i-1}$ moves out of its osculating plane in the same way as the vector L_{i+1} moves out of the plane Π_{i-1}.

• (since $\tau_i(t_{i-1})\triangle_{i-1} < 0$) the curve at $t = t_{i-1}$ moves out of its osculating plane opposite to the way as the vector L_{i-2} moves into the plane Π_{i-1} or L_i moves out of the plane Π_{i-2}.

Since in the definition 9.3, we have $\tau_i(t_i)\triangle_i > 0$, above analysis holds true when the curve is traversed in the reverse direction by concentrating the view on x_i instead of $\mathbf{x}_{i-1}$ (with set of points involved being $\{\mathbf{x}_{i-2}, \mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}, \mathbf{x}_{i+2}\}$)

Thus from the above analysis, we see that the definition 9.3 has better influence on the shape of the curve than that of the definition 9.2. However, the definition 9.2 helped us to bring a very important property (discussed above) of curve satisfying condition of definition 9.3.

10 Coplanarity preservation criteria for interpolating spline

Definition 10.1 *[9, Karavelas and Kaklis, 2000] Coplanarity preservation criteria is defined by the condition that if $\triangle_i = 0$ and $|N_{i-1}||N_i| \neq 0$, then*

$$\frac{|\omega(t) \times N_j|}{|\omega(t)||N_j|} < \epsilon_1, \, |\omega(t)| \neq 0, \, t \in I_i, \, j = i-1, i \tag{10.1}$$

where ϵ_1 is a user specified small positive number in $(0, 1]$, and I_i is user-specified closed interval such that $[t_{i-1}, t_i] \subseteq I_i \subseteq (t_{i-2}, t_{i+1})$.

Coplanarity preservation criteria states that if data points $\mathbf{x}_{i-2}$, $\mathbf{x}_{i-1}$, $\mathbf{x}_i$ and $\mathbf{x}_{i+1}$ are coplanar to a plane Π, then the interpolating curve between $\mathbf{x}_{i-1}$ and $\mathbf{x}_i$ and in the vicinity of $\mathbf{x}_{i-1}$ and $\mathbf{x}_i$ has its binormal close to N_j, that is, its osculating plane should remain close to a plane parallel to Π_j.

We observe that in addtion to the condition (10.1), the curve segment between $\mathbf{x}_{i-1}$ and $\mathbf{x}_i$ should be constrained such that its oscillations about the plane Π_j is minimum. In fact,

- if $|\omega(t_{i-1}) \times N_{i-1}| = 0$ and $|\omega(t_i) \times N_i| = 0$, then the curve segment can be constrained to be coplanar with the plane Π_i.
- if $|\omega(t_{i-1}) \times N_{i-1}| \neq 0$ and $|\omega(t_i) \times N_i| = 0$ (or $|\omega(t_{i-1}) \times N_{i-1}| = 0$ and $|\omega(t_i) \times N_i| \neq 0$) then the curve segment can be constrained such that it oscillation about the plane Π_i only once.

and if $|\omega(t_{i-1}) \times N_{i-1}| \neq 0$ and $|\omega(t_i) \times N_i| \neq 0$

- and addtionally N_{i-1} and N_i lie on the same side of the plane Π_j, then the curve segment can be constrained such that it does not oscillate about a (fixed) plane parallel to the plane Π_i and
- and addtionally when N_{i-1} and N_i lie on the opposite side of the plane Π_j, the curve segment can be constrained such that it oscillates about a (fixed) plane parallel to the plane Π_i only once.

11 Different shape preservation criteria on a curve segment

We now observe that the data points $\{\mathbf{x}_{i-2}, \mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}\}$ must satisfy one of the two conditions

c1 $N_{i-1} \cdot N_i < 0$, qualifying condition for inflection preservation criteria,

c2 $N_{i-1} \cdot N_i > 0$, qualifying condition for convexity preservation criteria

with one of the two conditions

t1 $\triangle_i = 0$, qualifying condition for torsion preservation criteria,

t2 $\triangle_i \neq 0$, qualifying condition for coplanarity preservation criteria.

We observe that there is no conflict between the conditions that the curve needs to satisfy for one among convexity preservation criteria and inflection preservation criteria simultaneously with one among torsion preservation criteria and coplanarity preservation criteria.

We also observe that there is no conflict between condition that curve need to satisfy for collinearity preservation criteria simultaneously with torsion preservation criteria or coplanarity preservation criteria.

12 Avoiding conflict between shape preservation behaviour of adjacent curve segments

We observed in previous section that every curve segment need to satisfy 1) condition for either convexity preservation criteria or inflection preservation criteria 2) condition for either torsion preservation criteria or coplanarity preservation criteria. In this section we investigate the compatibility between the shape preservation behaviour of adjacent curve segments. In Figure 19a-19d we observe that if the curve $\gamma(t)$ is required to be C^1 smooth then there is possibility that convexity as well as inflection preservation of a curve segment may lead to the violation of convexity and inflection preservation of adjacent curve segment.

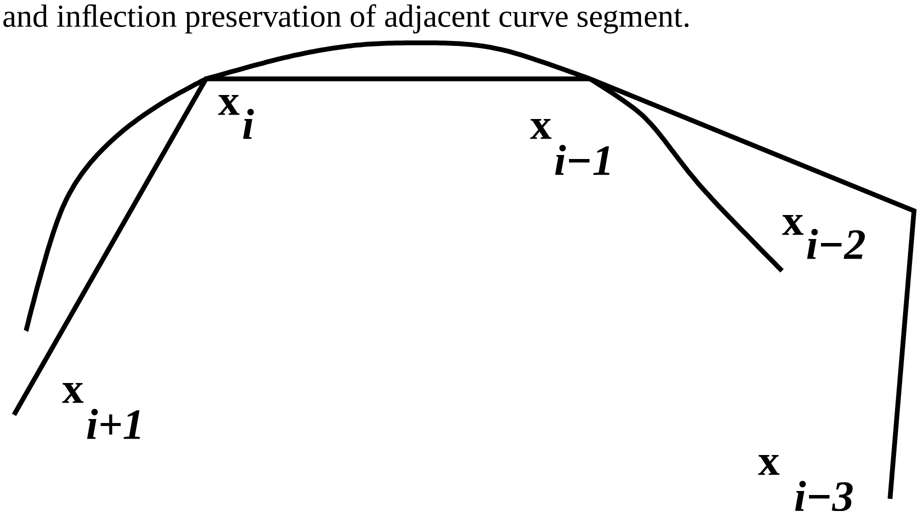


Figure 19a : Convexity preserving curve segment between $\mathbf{x}_{i-1}$ and $\mathbf{x}_i$ making violation of inflection preservation criteria between $\mathbf{x}_{i-1}$ and $\mathbf{x}_{i-2}$ imminent.

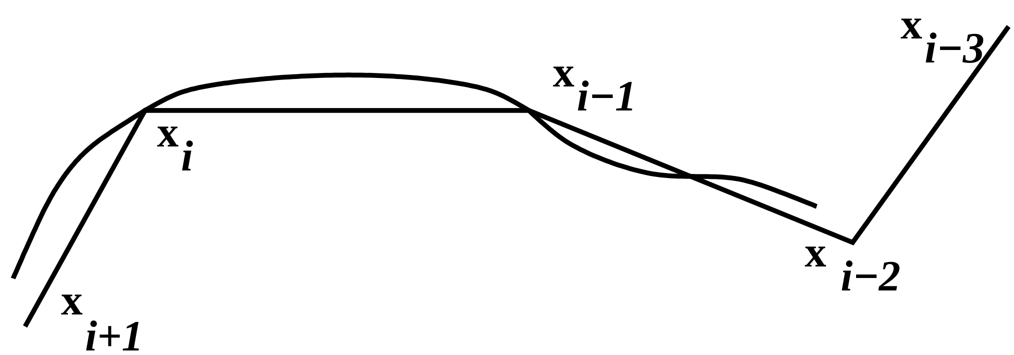


Figure 19b : Convexity preserving curve segment between $\mathbf{x}_{i-1}$ and $\mathbf{x}_i$ making violation of inflection preservation criteria between $\mathbf{x}_{i-1}$ and $\mathbf{x}_{i-2}$ imminent.

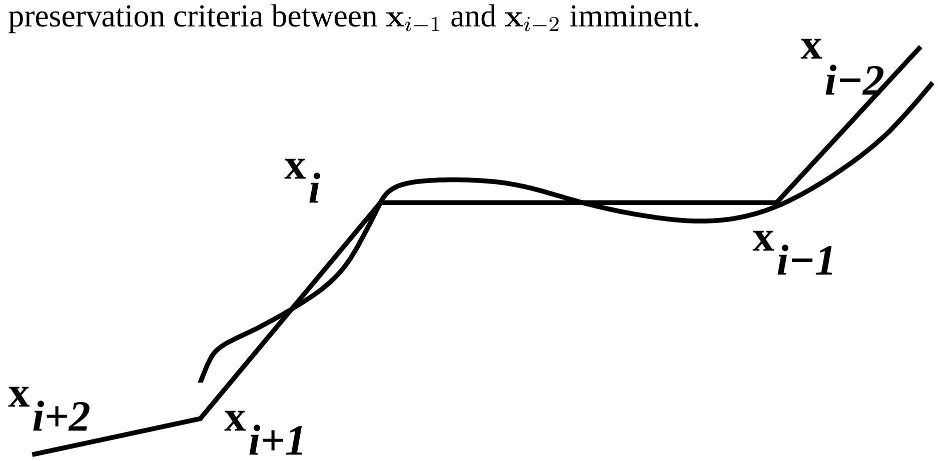


Figure 19c : Inflection preserving curve segment between $\mathbf{x}_{i-1}$ and $\mathbf{x}_i$ making violation of inflection preservation criteria between $\mathbf{x}_i$ and $\mathbf{x}_{i+1}$ imminent.

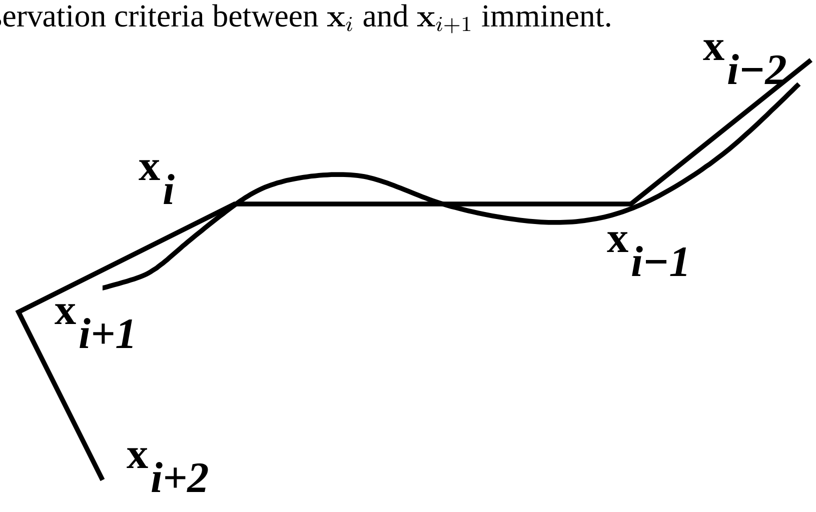


Figure 19d : Inflection preserving curve segment between $\mathbf{x}_{i-1}$ and $\mathbf{x}_i$ making violation of convexity preservation criteria between $\mathbf{x}_i$ and $\mathbf{x}_{i+1}$ imminent.

We observe from Figure 19a-19d that if $\mathbf{x}_{i-2}$, $\mathbf{x}_{i-1}$, $\mathbf{x}_i$, $\mathbf{x}_{i+1}$ are coplanar, then the conflict of convexity preservation criteria or inflection preservation of a curve segment with that of adjacent curve segment (of C^1 smooth spline curve $\gamma(t)$) is resolved if and only if $(\gamma'(t_{i-1}) \times L_i) \cdot$

$(\gamma'(t_{i-1}) \times L_{i-1}) < 0$.

For the general case, we observe from the definition of convexity and inflection preservation criteria that compatibility between convexity and inflection preservation behaviour of adjacent curve segments of C^1 smooth spline curve $\gamma(t)$ can be guaranteed if and only if

$$(\gamma'_{N_{i-1}^{\perp}}(t_{i-1}) \times L_i) \cdot (\gamma'_{N_{i-1}^{\perp}}(t_{i-1}) \times L_{i-1}) < 0 \tag{12.1}$$

where $\gamma_{N_i^{\perp}}(t)$ is the orthogonal projection the curve $\gamma(t)$ on a plane having normal vector as N_i. We note that the condition (12.1) does not interfere with the conditions of convexity and inflection preservation criteria.

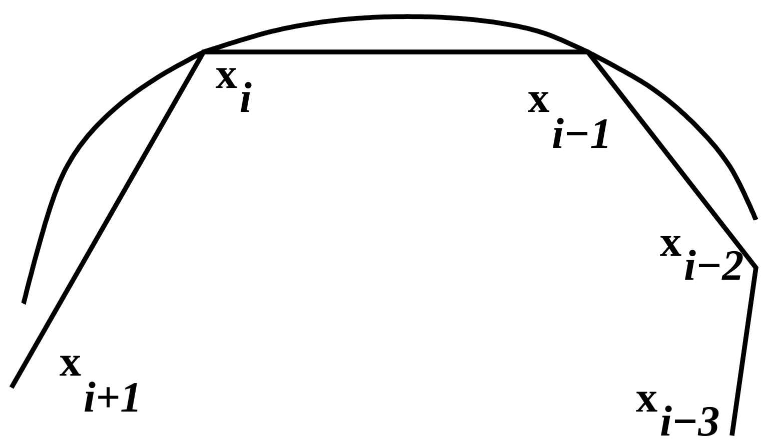


Figure 20a : Convexity preserving curve segment between $\mathbf{x}_{i-1}$ and $\mathbf{x}_i$ satsifying (12.1) facilitates inflection preservation between $\mathbf{x}_{i-1}$ and $\mathbf{x}_{i-2}$.

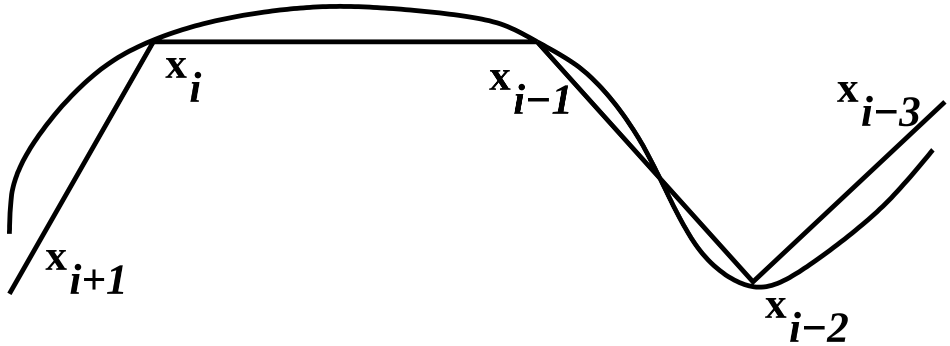


Figure 20b : Convexity preserving curve segment between $\mathbf{x}_{i-1}$ and $\mathbf{x}_i$ satsifying (12.1) facilitates inflection preservation between $\mathbf{x}_{i-1}$ and $\mathbf{x}_{i-2}$.

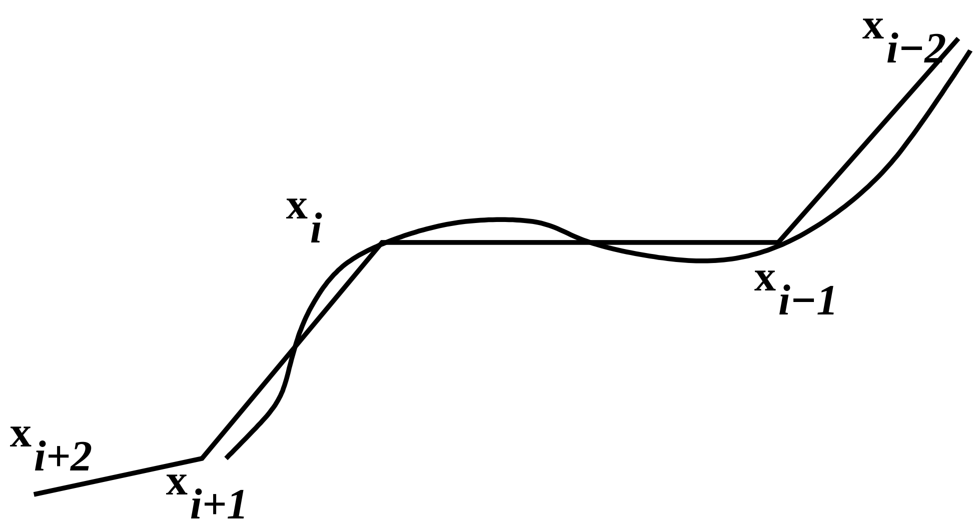


Figure 20c : Inflection preserving curve segment between $\mathbf{x}_{i-1}$ and $\mathbf{x}_i$ satsifying (12.1) facilitates inflection preservation between $\mathbf{x}_i$ and $\mathbf{x}_{i+1}$.

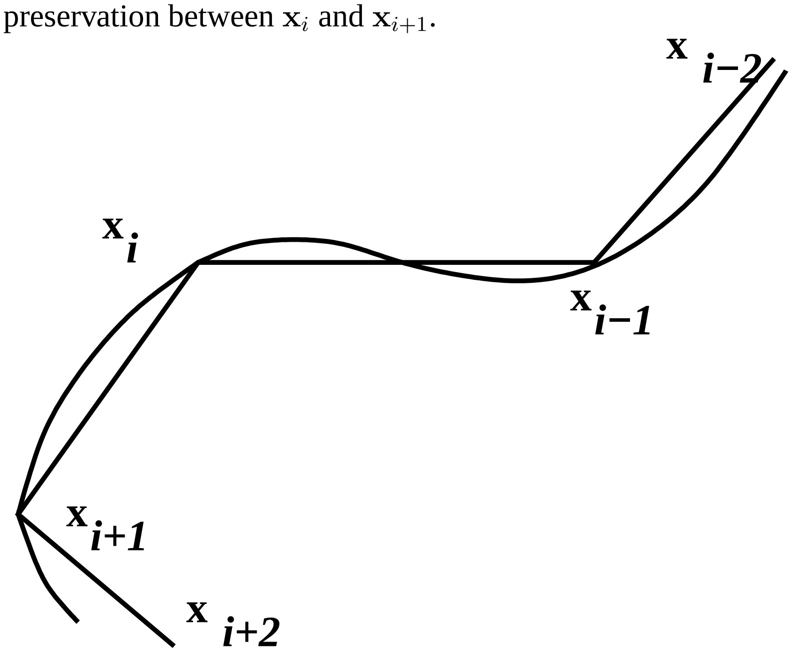


Figure 20d : Inflection preserving curve segment between $\mathbf{x}_{i-1}$ and $\mathbf{x}_i$ satsifying (12.1) facilitates convexity preservation between $\mathbf{x}_i$ and $\mathbf{x}_{i+1}$.

We also observe that the condition (12.1) can be conveniently imposed on a curve along with the conditions of torsion and coplanarity preservation criteria. Therefore, compatibility of torsion and coplanarity preservation of a curve segment with the convexity and inflection preservation of adjacent curve segment is feasible. From the above analysis and conditions of collinearity preservation criteria, we see that compatibility of convexity, inflection, torsion and coplanarity preservation of a curve segment with the collinearity preservation of adjacent curve

segment is also feasible.

We now analyze the compatibility between the adjacent curve segments satisfying coplanarity and torsion preservation criteria. Let us consider figure 18. The points $\mathbf{x}_{i-3}$, $\mathbf{x}_{i-2}$, $\mathbf{x}_{i-1}$, $\mathbf{x}_i$ and $\mathbf{x}_{i+1}$ may be such that 1) $\triangle_i\triangle_{i-1} < 0$, 2) $\triangle_i\triangle_{i-1} > 0$ and 3) $\triangle_i \neq 0$, $\triangle_{i-1} = 0$.

Theorem 12.1 *If $\triangle_i\triangle_{i-1} < 0$, then the curve segments $\gamma_{i-1}(t)$ (between $\mathbf{x}_{i-2}$ and $\mathbf{x}_{i-1}$) and $\gamma_i(t)$ (between $\mathbf{x}_{i-1}$ and $\mathbf{x}_i$), of spline curve $\gamma(t)$, satisfies conditions for torsion preservation criteria, if and only if either $\gamma(t)$ is torsion discontinuous or $\tau(t_{i-1}) = 0$ (and therefore $\tau_i(t_{i-1})\triangle_i = 0$, $\tau_{i-1}(t_{i-1})\triangle_{i-1} = 0$).*

Proof: As discussed in section 9 we have sign$(\tau_i(t_{i-1})\triangle_i)$ = sign$(\tau_i(t_{i-1})\triangle_{i-1}\triangle_i\triangle_{i-1})$. Therefore, from the definition 9.3 we see that torsion preservation by $\gamma_i(t)$ requires $\tau_i(t_{i-1})\triangle_{i-1} \leq 0$ and torsion preservation by $\gamma_{i-1}(t)$ requires $\tau_{i-1}(t_{i-1})\triangle_{i-1} \geq 0$. Hence the theorem. ∎

From the above proof it is evident that if $\triangle_i\triangle_{i-1} > 0$, then torsion preservation by γ_{i-1} and γ_i is compatible.

If $\triangle_i \neq 0$ and $\triangle_{i-1} = 0$ (requiring coplanarity preservation by $\gamma_{i-1}(t)$), then from the definition 10.1 we see that $\gamma_i(t)$ need to satisfy the condition of coplanarity condition 10.1, that is, binormal of the curve should be close to N_{i-1} in addition to satisfying torsion preservation criteria, for $t \in [t_{i-1}, t_i] \cap I_{i-1}$, where $[t_{i-2}, t_{i-1}] \subseteq I_{i-1} \subseteq (t_{i-3}, t_i)$.

13 Shape preservation by cubic interpolating splines

13.1 Cubic Bézier segments

Let the control polygon for Bézier representation of i^{th} cubic curve segment $\gamma_i(t)$ of cubic spline be $\{\mathbf{P}_{i,0}, \mathbf{P}_{i,1}, \mathbf{P}_{i,2}, \mathbf{P}_{i,3}\}$, with $\mathbf{P}_{i,0} = \mathbf{x}_i$ and $\mathbf{P}_{i,3} = \mathbf{x}_{i+1}$. For the i^{th} cubic curve segment $\gamma_i(t)$, of the cubic spline $\gamma(t)$ we have

$$\gamma_i(t) = \mathbf{P}_{i,0}B_0^3(u(t)) + \mathbf{P}_{i,1}B_1^3(u(t)) + \mathbf{P}_{i,2}B_2^3(u(t)) + \mathbf{P}_{i,3}B_3^3(u(t)) \qquad (13.1)$$

where $B_i^n(t)\ t \in [0,1]$ is i^{th} Bernstein's polynomial of order n, $u(t) = \dfrac{t - t_{i-1}}{t_i - t_{i-1}}$. Conditions for shape preservation criteria consists of first, second and third order derivatives of Bézier curves.

Therefore we get their expressions in terms of the end point and slopes at end points.

$$\gamma_i'(t) = \frac{3}{h_i}((\mathbf{P}_{i,1} - \mathbf{P}_{i,0})B_0^2(u(t)) + (\mathbf{P}_{i,2} - \mathbf{P}_{i,1})B_1^2(u(t)) + (\mathbf{P}_{i,3} - \mathbf{P}_{i,2})B_2^2(u(t))) \tag{13.2}$$

$$\gamma_i''(t) = \frac{6}{h_i^2}((\mathbf{P}_{i,2} - 2\mathbf{P}_{i,1} + \mathbf{P}_{i,0})(1 - u(t)) + (\mathbf{P}_{i,3} - 2\mathbf{P}_{i,2} + \mathbf{P}_{i,1})u(t)) \tag{13.3}$$

$$\gamma_i'''(t) = \frac{6}{h_i^3}(\mathbf{P}_{i,3} - 3\mathbf{P}_{i,2} + 3\mathbf{P}_{i,1} - \mathbf{P}_{i,0}) \tag{13.4}$$

We have $\mathbf{m}_{i-1} = \dfrac{3(\mathbf{P}_{i,1} - \mathbf{P}_{i,0})}{h_i}$, $\mathbf{m}_i = \dfrac{3(\mathbf{P}_{i,3} - \mathbf{P}_{i,2})}{h_i}$ and $L_i = \mathbf{P}_{i,3} - \mathbf{P}_{i,0}$. Therefore we can rewrite the expression for curve and its derivatives as follows

$$\begin{aligned}\gamma_i(t) &= \mathbf{x}_{i-1}B_0^3(u(t)) + (\mathbf{x}_{i-1} + \frac{h_i}{3}\mathbf{m}_{i-1})B_1^3(u(t)) + \\ &\quad (\mathbf{x}_i - \frac{h_i}{3}\mathbf{m}_i)B_2^3(u(t)) + \mathbf{x}_iB_3^3(u(t)),\end{aligned} \tag{13.5}$$

$$\gamma_i'(t) = \mathbf{m}_{i-1}B_0^2(u(t)) + (\frac{3}{h_i}L_i - \mathbf{m}_{i-1} - \mathbf{m}_i)B_1^2(u(t)) + \mathbf{m}_iB_2^2(u(t)), \tag{13.6}$$

$$\gamma_i''(t) = \frac{2}{h_i}((\frac{3}{h_i}L_i - 2\mathbf{m}_{i-1} - \mathbf{m}_i))(1 - u(t)) + (-\frac{3}{h_i}L_i + \mathbf{m}_{i-1} + 2\mathbf{m}_i)u(t)) \tag{13.7}$$

$$\gamma_i'''(t) = \frac{6}{h_i^3}(h_i(\mathbf{m}_{i-1} + \mathbf{m}_i) - 2L_i). \tag{13.8}$$

13.2 Convexity preservation criteria for cubic interpolating splines

Recall from theorem **??** that cubic spline $\gamma(t)$ interpolating data points $\mathbf{x}_i$, $0 \le i \le n$, satisfies convexity criteria if and only if the control polygons of the projection of $\gamma(t)$, $t \in [t_{i-1}, t_i]$ on planes with normal vectors N_j, $j = i-1, i$ are globally convex, whenever $N_{i-1} \cdot N_i \ge 0$.
We now find simplification of the condition of global convexity control polygon of $P_{N^\perp}(\gamma(t))$, $t \in [t_{i-1}, t_i]$. Using this simplified condition we find the modified convexity preservation criteria for cubic splines in theorem 13.4.

Lemma 13.1 *[3, Goldman, 1990] Let the points $\mathbf{P}_0$, $\mathbf{P}_1$ $\mathbf{P}_2$, $\mathbf{P}_3$ in R^3 lie on plane with normal vector $\mathbf{n}$. Then a line, through the points $\mathbf{P}_0$ and $\mathbf{P}_1$, intersects with a line, through the points $\mathbf{P}_2$ and $\mathbf{P}_3$, at the point*

$$\begin{aligned}\mathbf{P} = \;& \mathbf{P}_0 + (\mathbf{P}_1 - \mathbf{P}_0)s = \mathbf{P}_3 + (\mathbf{P}_2 - \mathbf{P}_3)t \\ = \;& \mathbf{P}_1 + (\mathbf{P}_0 - \mathbf{P}_1)\overline{s} = \mathbf{P}_2 + (\mathbf{P}_3 - \mathbf{P}_2)\overline{t}\end{aligned}$$

$$s = \frac{(\mathbf{P}_3 - \mathbf{P}_0) \times (\mathbf{P}_2 - \mathbf{P}_3) \cdot \mathbf{n}}{(\mathbf{P}_1 - \mathbf{P}_0) \times (\mathbf{P}_2 - \mathbf{P}_3) \cdot \mathbf{n}}, t = -\frac{(\mathbf{P}_1 - \mathbf{P}_0) \times (\mathbf{P}_3 - \mathbf{P}_0) \cdot \mathbf{n}}{((\mathbf{P}_1 - \mathbf{P}_0) \times (\mathbf{P}_2 - \mathbf{P}_3)) \cdot \mathbf{n}}$$

$$\overline{s} = \frac{(\mathbf{P}_2 - \mathbf{P}_1) \times (\mathbf{P}_3 - \mathbf{P}_2) \cdot \mathbf{n}}{(\mathbf{P}_0 - \mathbf{P}_1) \times (\mathbf{P}_3 - \mathbf{P}_2) \cdot \mathbf{n}}, \overline{t} = -\frac{(\mathbf{P}_0 - \mathbf{P}_1) \times (\mathbf{P}_2 - \mathbf{P}_1) \cdot \mathbf{n}}{(\mathbf{P}_0 - \mathbf{P}_1) \times (\mathbf{P}_3 - \mathbf{P}_2) \cdot \mathbf{n}}$$

Proof: To find the point of intersection the given two lines we need to solve the equations

$$\mathbf{P} = \mathbf{P}_0 + (\mathbf{P}_1 - \mathbf{P}_0)s \tag{13.1}$$

$$\mathbf{P} = \mathbf{P}_3 + (\mathbf{P}_2 - \mathbf{P}_3)t \tag{13.2}$$

for s and t with the condition for the coplanarity of the four points

$$(\mathbf{P}_3 - \mathbf{P}_0) \cdot ((\mathbf{P}_1 - \mathbf{P}_0) \times (\mathbf{P}_2 - \mathbf{P}_3)) = 0$$

By subtracting equation (13.1) from equation (13.2) we get

$$(\mathbf{P}_3 - \mathbf{P}_0) + (\mathbf{P}_2 - \mathbf{P}_3)t - (\mathbf{P}_1 - \mathbf{P}_0)s = 0 \tag{13.3}$$

Taking the cross product on both sides of equation (13.3) by $(\mathbf{P}_2 - \mathbf{P}_3)$ we get

$$(\mathbf{P}_3 - \mathbf{P}_0) \times (\mathbf{P}_2 - \mathbf{P}_3) = ((\mathbf{P}_1 - \mathbf{P}_0) \times (\mathbf{P}_2 - \mathbf{P}_3))s \tag{13.4}$$

Now taking the scalar product on both sides of equation (13.4) with **n** we get

$$s = \frac{((\mathbf{P}_3 - \mathbf{P}_0) \times (\mathbf{P}_2 - \mathbf{P}_3)) \cdot \mathbf{n}}{((\mathbf{P}_1 - \mathbf{P}_0) \times (\mathbf{P}_2 - \mathbf{P}_3)) \cdot \mathbf{n}} \tag{13.5}$$

Similarly we get the value for t. Now by interchanging $\mathbf{P}_0$ with $\mathbf{P}_1$ and $\mathbf{P}_3$ with $\mathbf{P}_2$ we get the values for $\overline{s}$ and $\overline{t}$. Hence proved. ∎

Lemma 13.2 *A planar polygonal arc* $\{\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3\}$ *lying on a plane with normal vector* N *is globally convex according to orientation induced by* N *if and only if either*

1. $(\mathbf{P}_1 - \mathbf{P}_0) \times (\mathbf{P}_2 - \mathbf{P}_3) \cdot N > 0$ *with*

   (a) $(\mathbf{P}_1 - \mathbf{P}_0) \times (\mathbf{P}_2 - \mathbf{P}_1) \cdot N < 0$ *and* $(\mathbf{P}_2 - \mathbf{P}_1) \times (\mathbf{P}_3 - \mathbf{P}_2) \cdot N < 0$ *or*

   (b) $(\mathbf{P}_0 - \mathbf{P}_1) \times (\mathbf{P}_3 - \mathbf{P}_0) \cdot N < 0$ *and* $(\mathbf{P}_3 - \mathbf{P}_0) \times (\mathbf{P}_2 - \mathbf{P}_3) \cdot N < 0$

   or

2. $(\mathbf{P}_1 - \mathbf{P}_0) \times (\mathbf{P}_2 - \mathbf{P}_3) \cdot N < 0$ *with*

   (a) $(\mathbf{P}_1 - \mathbf{P}_0) \times (\mathbf{P}_2 - \mathbf{P}_1) \cdot N > 0$ *and* $(\mathbf{P}_2 - \mathbf{P}_1) \times (\mathbf{P}_3 - \mathbf{P}_2) \cdot N > 0$ *or*

   (b) $(\mathbf{P}_0 - \mathbf{P}_1) \times (\mathbf{P}_3 - \mathbf{P}_0) \cdot N > 0$ *and* $(\mathbf{P}_3 - \mathbf{P}_0) \times (\mathbf{P}_2 - \mathbf{P}_3) \cdot N > 0$

holds.

Proof: From the definition 3.4 we know that planar polygonal arc $\{\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3\}$ is globally convex according to the orientation induced by normal vector N if and only if

condition i polygonal arc starting from $\mathbf{P}_0$ always turn towards the right side and

condition ii it always lies entirely to its right side of any of its edges.

For the given polygonal arc, **condition i** holds if and only if

$$((\mathbf{P}_1 - \mathbf{P}_0) \times (\mathbf{P}_2 - \mathbf{P}_1)) \cdot N((\mathbf{P}_2 - \mathbf{P}_1) \times (\mathbf{P}_3 - \mathbf{P}_2)) \cdot N > 0 \tag{13.6}$$

Now for the given polygonal arc, **condition ii** holds if and only if

1. line through $\mathbf{P}_0$ and $\mathbf{P}_1$ does not intersect the line segment between $\mathbf{P}_2$ and $\mathbf{P}_3$ and
2. line through $\mathbf{P}_2$ and $\mathbf{P}_3$ does not intersect the line segment between $\mathbf{P}_1$ and $\mathbf{P}_0$.

The above two condition holds if and only if the point of intersection $\mathbf{P}$, between the line l_1 through $\mathbf{P}_0$ and $\mathbf{P}_1$ and line l_2 through $\mathbf{P}_2$ and $\mathbf{P}_3$, does not lie in the segment between $\mathbf{P}_0$ and $\mathbf{P}_1$ or the segment between $\mathbf{P}_2$ and $\mathbf{P}_3$. From lemma 13.1 we know that point of intersection of lines l_1 and l_2 is given by

$$\begin{aligned} \mathbf{P} = \; & \mathbf{P}_0 + (\mathbf{P}_1 - \mathbf{P}_0)s = \; \mathbf{P}_3 + (\mathbf{P}_2 - \mathbf{P}_3)t \\ = \; & \mathbf{P}_1 + (\mathbf{P}_0 - \mathbf{P}_1)\overline{s} = \; \mathbf{P}_2 + (\mathbf{P}_3 - \mathbf{P}_2)\overline{t} \end{aligned}$$

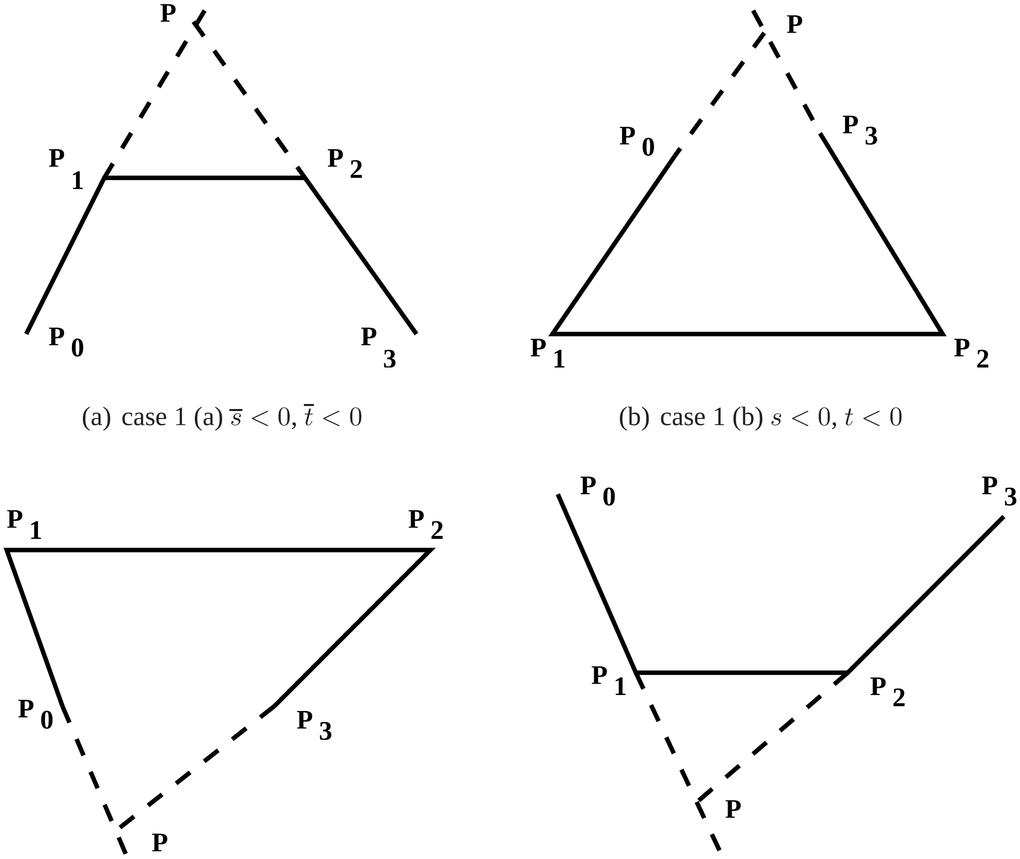
$$s = \frac{((\mathbf{P}_3 - \mathbf{P}_0) \times (\mathbf{P}_2 - \mathbf{P}_3)) \cdot \mathbf{N}}{((\mathbf{P}_1 - \mathbf{P}_0) \times (\mathbf{P}_2 - \mathbf{P}_3)) \cdot \mathbf{N}}, \; t = -\frac{((\mathbf{P}_1 - \mathbf{P}_0) \times (\mathbf{P}_3 - \mathbf{P}_0)) \cdot \mathbf{N}}{((\mathbf{P}_1 - \mathbf{P}_0) \times (\mathbf{P}_2 - \mathbf{P}_3)) \cdot \mathbf{N}}$$

$$\overline{s} = \frac{((\mathbf{P}_2 - \mathbf{P}_1) \times (\mathbf{P}_3 - \mathbf{P}_2)) \cdot \mathbf{N}}{((\mathbf{P}_0 - \mathbf{P}_1) \times (\mathbf{P}_3 - \mathbf{P}_2)) \cdot \mathbf{N}}, \; \overline{t} = -\frac{((\mathbf{P}_0 - \mathbf{P}_1) \times (\mathbf{P}_2 - \mathbf{P}_1)) \cdot \mathbf{N}}{((\mathbf{P}_0 - \mathbf{P}_1) \times (\mathbf{P}_3 - \mathbf{P}_2)) \cdot \mathbf{N}}$$

Now with **condition i** ensured, **condition ii** is satisfied if and only if either $s < 0$ with $t < 0$ or $\overline{s} < 0$ with $\overline{t} < 0$ holds. Hence proved. ∎

Lemma 13.3 *Let* $\overline{\mathbf{P}}_i = P_{N^\perp}(\mathbf{P}_i)$. *Then*

$$(\overline{\mathbf{P}}_a - \overline{\mathbf{P}}_b) \times (\overline{\mathbf{P}}_c - \overline{\mathbf{P}}_d) \cdot N \;\; = \;\; (\mathbf{P}_a - \mathbf{P}_b) \times (\mathbf{P}_c - \mathbf{P}_d) \cdot N \tag{13.7}$$



(a) case 1 (a) $\overline{s} < 0, \overline{t} < 0$ (b) case 1 (b) $s < 0, t < 0$

(c) case 2 (a) $\overline{s} < 0, \overline{t} < 0$ (d) case 2 (b) $s < 0, t < 0$

Figure 21: Examples of different planar convex control polygons

Proof: We know that projection of $\mathbf{P}_i$ on a plane $\dfrac{(x, y, z) \cdot N + d}{\|N\|} = 0$ with normal vector N, $P_{N^\perp}(\mathbf{P}_i) = \overline{\mathbf{P}}_i$ is given by

$$\overline{\mathbf{P}}_i = \mathbf{P}_i + \frac{\mathbf{P}_i \cdot N + d}{\|N\|^2} N. \tag{13.8}$$

One can get the proof using the idea in the proof of lemma 3.13. ∎

Using the definition 3.16 and following theorem from [11, Liu, 2001] we get

Theorem 13.4 *A cubic spline curve $\gamma(t)$ satisfies the convexity preservation criteria if and only if either*

1. $\mathbf{m}_{i-1} \times \mathbf{m}_i \cdot N_j < 0$, *with*

   (a) $\mathbf{m}_{i-1} \times L_i \cdot N_j < \dfrac{h_i}{3} \mathbf{m}_{i-1} \times \mathbf{m}_i \cdot N_j$ *and* $L_i \times \mathbf{m}_i \cdot N_j < \dfrac{h_i}{3} \mathbf{m}_{i-1} \times \mathbf{m}_i \cdot N_j$ *or*

   (b) $\mathbf{m}_{i-1} \times L_i \cdot N_j > 0$ *and* $L_i \times \mathbf{m}_i \cdot N_j > 0$

or

2. $\mathbf{m}_{i-1} \times \mathbf{m}_i \cdot N_j > 0$*, with*

   (a) $\mathbf{m}_{i-1} \times L_i \cdot N_j > \dfrac{h_i}{3}\mathbf{m}_{i-1} \times \mathbf{m}_i \cdot N_j$ *and* $L_i \times \mathbf{m}_i \cdot N_j > \dfrac{h_i}{3}\mathbf{m}_{i-1} \times \mathbf{m}_i \cdot N_j$ *or*

   (b) $\mathbf{m}_{i-1} \times L_i \cdot N_j < 0$ *and* $L_i \times \mathbf{m}_i \cdot N_j < 0$

$j = i-1, i$*, whenever* $N_{i-1} \cdot N_i > 0$*.*

Proof: We note that

$$(\mathbf{P}_1 - \mathbf{P}_0) \times (\mathbf{P}_2 - \mathbf{P}_1) \cdot N = (\mathbf{P}_1 - \mathbf{P}_0) \times (\mathbf{P}_3 - \mathbf{P}_0) \cdot N + (\mathbf{P}_1 - \mathbf{P}_0) \times (\mathbf{P}_2 - \mathbf{P}_3) \cdot N$$
$$(\mathbf{P}_2 - \mathbf{P}_1) \times (\mathbf{P}_3 - \mathbf{P}_2) \cdot N = (\mathbf{P}_3 - \mathbf{P}_0) \times (\mathbf{P}_3 - \mathbf{P}_2) \cdot N - (\mathbf{P}_1 - \mathbf{P}_0) \times (\mathbf{P}_3 - \mathbf{P}_2) \cdot N$$

since $\mathbf{P}_2 - \mathbf{P}_1 = (\mathbf{P}_2 - \mathbf{P}_3) - (\mathbf{P}_1 - \mathbf{P}_0) + (\mathbf{P}_3 - \mathbf{P}_0)$

Since control polygon of (orthogonal) projection of Bézier curve on a plane is same as the orthogonal projection of control polygon of the curve on the plane, using theorem **??**, lemma 13.2 (by replacing $\mathbf{P}_i$ by $\mathbf{P}_{i,j}$) and lemma 13.3 we get the theorem. ∎

13.3 Inflection preservation criteria for cubic interpolating splines

The definition 6.2 of inflection preservation criteria involves the curvature term $\omega_i(t)$. So, in order to get a simplified characterization for inflection criteria for cubic case we first get simplified expression for $\omega_i(t)$ as follows.

Lemma 13.5 *Let* $\mathbf{c}(t) = \mathbf{P}_0(1-t)^2 + \mathbf{P}_1(2\,t\,(1-t)) + \mathbf{P}_2 t^2$ *be a quadratic Bézier curve. Then*

$$\mathbf{c}(t) \times \mathbf{c}'(t) = 2(\mathbf{P}_0 \times \mathbf{P}_1)(1-t)^2 + (\mathbf{P}_0 \times \mathbf{P}_2)(2\,t\,(1-t)) + 2(\mathbf{P}_1 \times \mathbf{P}_2)t^2. \qquad (13.1)$$

Proof: In the statement of the lemma we observe that though the $\mathbf{c}(t)$ and $\mathbf{c}'(t)$ are Bézier curves of degree 2 and 1 respectively, their cross product is a Bézier curve of degree 2 instead of 3. In order to understand this we express the curve $\mathbf{c}(t)$ in power basis form as $\mathbf{c}(t) = \mathbf{p}_0 + \mathbf{p}_1 t + \mathbf{p}_2 t^2$, where $\mathbf{p}_0 = \mathbf{P}_0$, $\mathbf{p}_1 = 2(\mathbf{P}_1 - \mathbf{P}_0)$, $\mathbf{p}_2 = \mathbf{P}_0 - 2\mathbf{P}_1 + \mathbf{P}_2$. Now the $z-$coordinate of $(\mathbf{c}(t) \times \mathbf{c}'(t))$ is

$$\begin{vmatrix} p_{0,x} + p_{1,x}t + p_{2,x}t^2 & p_{0,y} + p_{1,y}t + p_{2,y}t^2 \\ p_{1,x} + 2p_{2,x}t & p_{1,y} + 2p_{2,y}t \end{vmatrix} = \begin{vmatrix} p_{1,x} & p_{1,y} \\ p_{2,x} & p_{2,y} \end{vmatrix} t^2 + 2\begin{vmatrix} p_{0,x} & p_{0,y} \\ p_{2,x} & p_{2,y} \end{vmatrix} t + \begin{vmatrix} p_{0,x} & p_{0,y} \\ p_{1,x} & p_{1,y} \end{vmatrix}$$

The coefficient of t^3 is 0 due to special relation between the curve and its derivative. (We study this phenomenon, in detail, in chapter **??**.) Thus we see that $\mathbf{c}(t) \times \mathbf{c}'(t) = (\mathbf{p}_1 \times \mathbf{p}_2)t^2 + (\mathbf{p}_0 \times \mathbf{p}_2)t + (\mathbf{p}_0 \times \mathbf{p}_1)$. On substituting the values of $\mathbf{p}_i$ we get the relation 13.1. ∎

Now by substituting the expression for $\gamma_i'(t)$ from section 13.1 we get the expression for curvature of cubic curve as

$$\begin{aligned} \omega_i(t) &= \gamma_i'(t) \times \gamma_i''(t) \\ &= \mathbf{g}_{0,i}(1-u(t))^2 + \mathbf{g}_{1,i}u(t)(1-u(t)) + \mathbf{g}_{2,i}(u(t))^2 \end{aligned} \tag{13.2}$$

where

$$\mathbf{g}_{0,i} = \frac{6}{h_i^2}(\mathbf{m}_{i-1} \times L_i) - \frac{2}{h_i}(\mathbf{m}_{i-1} \times \mathbf{m}_i) \tag{13.3}$$

$$\mathbf{g}_{1,i} = \frac{2}{h_i}(\mathbf{m}_{i-1} \times \mathbf{m}_i) \tag{13.4}$$

$$\mathbf{g}_{2,i} = \frac{6}{h_i^2}(L_i \times \mathbf{m}_i) - \frac{2}{h_i}(\mathbf{m}_{i-1} \times \mathbf{m}_i) \tag{13.5}$$

Theorem 13.6 *If the i^{th} curve segment of interpolating spline $\gamma_i(t)$ is a cubic curve, $\omega_i(t_j) \cdot N_j > 0$ and $\omega_i(t) \cdot N_j > 0$ changes sign only once for $t \in [t_{i-1}, t_i]$, $j = i-1, i$ then for all $N = \lambda N_{i-1} + \mu N_i$, where $\lambda\mu < 0$, $\omega_i(t) \cdot N$ has precisely one sign change for $t \in [t_{i-1}, t_i]$.*

Proof: We prove the theorem for the case $\lambda > 0$. We first note that $\lambda\omega_i(t_{i-1}) \cdot N_{i-1} > 0$ $\mu\omega_i(t_{i-1}) \cdot N_i > 0$. Since $\omega_i(t) \cdot N_j > 0$ changes sign only once for $t \in [t_{i-1}, t_i]$, $j = i-1, i$ we also have $\lambda\omega_i(t_i) \cdot N_{i-1} < 0$ $\mu\omega_i(t_i) \cdot N_i < 0$. Thus $\lambda\omega_i(t_{i-1}) \cdot N_{i-1} + \mu\omega_i(t_{i-1}) \cdot N_i = \omega_i(t_{i-1}) \cdot N > 0$ and $\lambda\omega_i(t_i) \cdot N_{i-1} + \mu\omega_i(t_i) \cdot N_i = \omega_i(t_i) \cdot N < 0$.

From formula (13.2) we see that $\omega_i(t)$ is a quadratic curve and hence $\omega_i(t) \cdot N$ is quadratic polynomial for $j = i-1, i$. That is, $\omega_i(t) \cdot N$ can change sign only twice. And if $\omega_i(t) \cdot N_{i-1}$ changes sign twice for $t \in [t_{i-1}, t_i]$, then we must have $\omega_i(t_{i-1}) \cdot N > 0$ and $\omega_i(t_i) \cdot N > 0$. Therefore $\omega_i(t) \cdot N_j$ changes sign only once for $t \in [t_{i-1}, t_i]$.

The proof for the case $\lambda < 0$ is similar. ∎

Remark 13.7 *In [7, Goodman and Ong, CAGD 15, 1-17, 1997], theorem 13.6 is proved for rational cubic Bézier curve for a special case in which the tangent vector $\mathbf{m}_j$ lies on the plane with normal vector as N_j, for $j = i-1, i$. But this is a heavy restriction for modeling curves and surfaces.*

Theorem 13.8 *If the i^{th} curve segment of interpolating spline $\gamma_i(t)$ is a cubic curve and $\omega_i(t_{i-1}) \cdot N_{i-1} > 0$, and $\omega_i(t_i) \cdot N_{i-1} < 0$ then $\omega_i(t) \cdot N_{i-1}$ changes sign only once for $t \in [t_{i-1}, t_i]$.*

Proof: From formula (13.2) we see that $\omega_i(t)$ is a quadratic curve and hence $\omega_i(t) \cdot N_j$ is quadratic polynomial for $j = i-1, i$. That is, $\omega_i(t) \cdot N_{i-1}$ (and $\omega_i(t) \cdot N_i$) can change sign only twice. And if $\omega_i(t) \cdot N_{i-1}$ changes sign twice for $t \in [t_{i-1}, t_i]$, then $\omega_i(t_{i-1}) \cdot N_{i-1}$ and $\omega_i(t_i) \cdot N_{i-1}$ will have same sign. But we have $\omega_i(t_{i-1}) \cdot N_{i-1} > 0$ and $\omega_i(t_i) \cdot N_{i-1} < 0$. Therefore $\omega_i(t) \cdot N_j$ changes sign only once for $t \in [t_{i-1}, t_i]$. ∎

Theorem 13.9 *If the i^{th} curve segment of interpolating spline $\gamma_i(t)$ is a cubic curve and $\omega_i(t_{i-1}) \cdot N_i < 0$, and $\omega_i(t_i) \cdot N_{i-1} > 0$ then $\omega_i(t) \cdot N_i$ changes sign only once for $t \in [t_{i-1}, t_i]$.*

Proof: The proof is similar to that of theorem 13.8. ∎

Remark 13.10 *The ease of using cubic curve, justified by theorems 13.6, 13.8 and 13.9, is one of the reasons for using cubic curve instead of higher order curves in splines.*

Theorem 13.11 *$\gamma_i(t)$ satisfies inflection criteria if and only if*

1. $\mathbf{g}_{0,i} \cdot N_{i-1} > 0$, $\mathbf{g}_{0,i} \cdot N_i < 0$ *and*

2. $\mathbf{g}_{2,i} \cdot N_{i-1} < 0$ $\mathbf{g}_{2,i} \cdot N_i > 0$

whenever $N_{i-1} \cdot N_i < 0$.

Proof: Proof follows from the conditions in the definition 6.2 for inflection criteria of splines, equation (13.2) and theorem 13.6, 13.8 and 13.9. ∎

Remark 13.12 *It is known that quadratic curves cannot be used in splines interpolating non-planar set of data points as it does not exhibit torsion. But now we can easily see that the quadratic curves cannot even be used in splines interpolating planar set of data points because it cannot satisfy the conditions for inflection preserving criteria.*

13.4 Torsion preservation criteria for cubic interpolating splines

We first state our result for torsion of a cubic Bézier curve. Using the identities in section 13.3 we get

Theorem 13.13 *Let $\overline{\tau}_i(t) = |\gamma_i'(t)\gamma_i''(t)\gamma_i'''(t)|$, $t \in [t_{i-1}, t_i]$ (numerator of $\tau_i(t)$). Then $\overline{\tau}_i(t) = \dfrac{12}{h_i^4}(\mathbf{m}_{i-1} \times \mathbf{m}_i \cdot L_i)$*

Proof: We can write numerator of $\tau_i(t)$ as

$$\begin{aligned}\overline{\tau}_i(t) &= |\gamma_i'(t)\gamma_i''(t)\gamma_i'''(t)| \\ &= \omega_i(t)\cdot\gamma_i'''(t) \\ &= h_{0,i}(1-u(t))^2 + h_{1,i}u(t)(1-u(t)) + h_{2,i}(u(t))^2\end{aligned}$$

where

$$\begin{aligned}h_{0,i} &= (\frac{6}{h_i^2}(\mathbf{m}_{i-1}\times L_i) - \frac{2}{h_i}(\mathbf{m}_{i-1}\times\mathbf{m}_i))\cdot\frac{6}{h_i^3}(h_i(\mathbf{m}_{i-1}+\mathbf{m}_i) - 2L_i) \\ &= \frac{6}{h_i^4}(6(\mathbf{m}_{i-1}\times L_i)\cdot\mathbf{m}_i + 4(\mathbf{m}_{i-1}\times\mathbf{m}_i)\cdot L_i) = \frac{12}{h_i^4}(\mathbf{m}_{i-1}\times L_i\cdot\mathbf{m}_i)\end{aligned}$$

$$h_{1,i} = \frac{2}{h_i}(\mathbf{m}_{i-1}\times\mathbf{m}_i)\cdot\frac{6}{h_i^3}(h_i(\mathbf{m}_{i-1}+\mathbf{m}_i) - 2L_i) = \frac{24}{h_i^4}(\mathbf{m}_{i-1}\times L_i\cdot\mathbf{m}_i)$$

$$\begin{aligned}h_{2,i} &= (\frac{6}{h_i^2}(L_i\times\mathbf{m}_i) - \frac{2}{h_i}(\mathbf{m}_{i-1}\times\mathbf{m}_i))\cdot\frac{6}{h_i^3}(h_i(\mathbf{m}_{i-1}+\mathbf{m}_i) - 2L_i) \\ &= \frac{6}{h_i^4}(6(L_i\times\mathbf{m}_i)\cdot\mathbf{m}_{i-1} + 4(\mathbf{m}_{i-1}\times\mathbf{m}_i)\cdot L_i) = \frac{12}{h_i^4}(\mathbf{m}_{i-1}\times L_i\cdot\mathbf{m}_i)\end{aligned}$$

$u(t) = \dfrac{t-t_{i-1}}{t_i - t_{i-1}}$, $t\in[t_{i-1},t_i]$. Therefore

$$\begin{aligned}\overline{\tau}_i(t) &= \frac{12}{h_i^4}((\mathbf{m}_{i-1}\times L_i\cdot\mathbf{m}_i)((1-u(t))^2 + 2u(t)(1-u(t)) + (u(t))^2) \\ &= \frac{12}{h_i^4}(\mathbf{m}_{i-1}\times L_i\cdot\mathbf{m}_i)\end{aligned}$$

Hence proved. ∎

From definition 9.3 of torsion preservation criteria and theorem 13.13 we get the following

Theorem 13.14 *A cubic spline curve $\gamma(t)$ satisfies torsion preservation criteria, for $t\in[t_{i-1},t_i]$, if and only if $[\mathbf{m}_{i-1}\ L_i\ \mathbf{m}_i]\triangle_i > 0$, whenever $\triangle_i \neq 0$.*

13.5 Collinearity preservation criteria for cubic interpolating spline

With the interpretation given in section 8 we get a sufficient condition for collinearity preservation criteria for splines in terms of Bézier control points of curve segments as below. Let the control points of curve segment $\gamma(t)$, $t\in[t_i,t_{i+1}]$ be $\mathbf{P}_{i,j}$, $j = 0,1,\ldots,m_i$ and the control polygon be represented as $\mathcal{P}_i$. The control points of the derivative of curve segment are $\mathbf{P}'_{i,j} = \dfrac{3}{h_i}(\mathbf{P}_{i,j+1} - \mathbf{P}_{i,j})$, $j = 0,\ldots,m_i-1$ and the control polygon be represented as $\mathcal{P}'_i$.

Now we make a observation about Bézier curves based on following interpretation about vector product between two vectors in R^3.

$$\frac{|\mathbf{A}\times\mathbf{B}|}{|\mathbf{A}||\mathbf{B}|} = \sin(\theta),\ \mathbf{A},\mathbf{B}\in R^3,\ \theta \text{ is the angle between } \mathbf{A} \text{ and } \mathbf{B} \tag{13.1}$$

For any vector $L \in R^3$ equation 13.1 implies the following

Lemma 13.15 *If* $\theta_i = \angle \mathbf{p}_i L \leq 90^o$, $i = 1, 2$, $\mathbf{p}_1$, $\mathbf{p}_2 \in R^3$, *then*

$$\sup\left\{\frac{|\mathbf{p}\times L|}{|\mathbf{p}||L|} : \mathbf{p} = \mathbf{p}_1(1-t) + \mathbf{p}_2 t,\ t\in[0,1]\right\} = \sup\left\{\frac{|\mathbf{p}_1\times L|}{|\mathbf{p}_1||L|}, \frac{|\mathbf{p}_2\times L|}{|\mathbf{p}_2||L|}\right\}.$$

Following lemma follows from lemma 13.15

Lemma 13.16 *If* $\theta_i = \angle \mathbf{p}_i L \leq 90^o$, $i = 1, 2, 3$, *then*

$$\sup\left\{\frac{|\mathbf{p}\times L|}{|\mathbf{p}||L|} : \mathbf{p} \textit{ is a point inside the planar triangle formed by the points } \mathbf{p}_1, \mathbf{p}_2 \textit{ and } \mathbf{p}_3 \in R^3\right\} = \sup\left\{\frac{|\mathbf{p}_i\times L|}{|\mathbf{p}_i||L|} : i = 1,2,3\right\}$$

We observe that

Theorem 13.17 *For a Bézier curve* $\mathbf{c}(t)$ *with control points* $\mathbf{p}_i$, $\theta_i = \angle \mathbf{p}_i L \leq 90^o$, $i = 0, 1, \ldots, m$.

$$\sup\left\{\frac{|\mathbf{c}(t)\times L|}{|\mathbf{c}(t)||L|} : t\in[0,1]\right\} < \sup\left\{\frac{|\mathbf{p}\times L|}{|\mathbf{p}_i||L|} : \mathbf{p} \textit{ belongs to set of vertices of convex hull formed by } \mathbf{p}_i i = 0,1,\ldots,m\right\} \tag{13.2}$$

Proof: Proof follows from equation (13.1) convex hull property of Bézier curves (which states that Bézier curves lie inside the convex hull, that is, smallest convex polyhedra, with triangular sides, formed by its control points) and lemma 13.16. ∎

Theorem 13.18 $\gamma(t)$ *satisfies collinearity criteria if*

$$\sup\left\{\frac{|\mathbf{P}'_{i,k}\times L_j|}{|\mathbf{P}'_{i,k}||L_j|} : k = 0,1,2\right\} < \epsilon,\ j = i-1, i, \tag{13.3}$$

whenever $|N_i| = 0$ *and* $L_{i-1}\cdot L_i > 0$, $0 < \theta_k = \angle \mathbf{P}'_{i,k} L_j \leq 90^o$, $k = 0,1,2$, $j = i-1, i$ *(a reasonable assumption to make).*

Proof: Since for a quadratic Bézier curve convex hull of its control points is the triangle formed by the control points, therefore

$$\sup\left\{\frac{|\gamma'(t)\times L_j|}{|\gamma'(t)||L_j|}:t\in[t_{i-1},t_i]\right\}<$$

$$\sup\left\{\frac{|\mathbf{P}'_{i,k}\times L_j|}{|\mathbf{P}'_{i,k}||L_j|}:k=0,1,2\right\},j=i-1,i, \tag{13.4}$$

Hence proved. ∎

In case in the definition 10.1 the condition 10.1 is replaced by $\omega_i(t)=0$, $t\in\eta_i$ as stated in papers [7, Goodman and Ong, CAGD 15, 1997] etc., the collinearity condition for cubic spline would have required $\mathbf{m}_i=\alpha L_i$, $\alpha>0$ in place of 13.3.

13.6 Coplanarity preservation criteria for interpolating cubic splines

From the definition of coplanarity criteria 10.1 and the analysis in the previous section we have following theorem stating the sufficient condition for cubic spline to satisfy co-planarity preservation criterion.

Theorem 13.19 *$\gamma_i(t)$ satisfies the co-planarity preservation criteria if*

$$\sup\left\{\frac{|\mathbf{g}_{k,i}\times N_j|}{|\mathbf{g}_{k,i}||N_j|}:k=0,1,2\right\}<\epsilon,j=i-1,i, \tag{13.1}$$

whenever $\triangle_i=0$ and $|N_{i-1}||N_i|\neq 0$ $\theta_k=\angle\mathbf{g}_{i,k}N_j\leq 90^o$, $k=0,1,2$, $j=i-1$ or i (a reasonable assumption to make).

Proof: Proof is same as the proof of theorem 13.18 ∎

In case in the definition 10.1 the condition 10.1 is replaced by $\tau_i(t)=0$, $t\in[t_{i-1},t_i]$ as stated in papers [7, Goodman and Ong, CAGD 15, 1997] etc., the coplanarity condition for cubic spline would have required $\mathbf{m}_{i-1}\times L_i\cdot\mathbf{m}_i=0$, that is, $\mathbf{m}_i=\alpha L_i+\beta\mathbf{m}_{i-1}$, $\alpha,\beta\in R$ in place of 13.1.

Alternatively (actually more precisely) the condition 13.1 can be replaced by the pair of conditions $m_{i-1}=\alpha_1L_i+\beta_1L_{i-1}$, $m_i=\alpha_2L_i+\beta_2L_{i+1}$, with α_1, α_2, β_1, $\beta_2>0$.

14 Shape preserving properties of cubic Catmull-Rom splines

Let denote the vector $\mathbf{x}_{i+1}-\mathbf{x}_{i-1}$ as $\mathbf{t}_i$, $i=2,...,n-1$, the plane containing the data points $\mathbf{x}_{i-1}$, $\mathbf{x}_i$, $\mathbf{x}_{i+1}$ as Π_i, $i=2,...,n-1$ and for a curve $\gamma(t)=[x(t),y(t),z(t)]$, $t\in[0,1]$ in R^3

let $\omega(t) = \gamma'(t) \times \gamma''(t)$. We now prove that cubic Catmull-Rom splines, with magnitude of tangent vectors calculated according to our algorithm, preserve convexity, inflection, torsion, collinearity and coplanarity behavior of the data polygon, as follows.

The tangent vector of the Catmull-Rom spline at a data point $\mathbf{x}_i$, $i = 2, ..., n-1$ is parallel to $\mathbf{t}_i$ so that the tangent vector is coplanar with the plane Π_i. Due to this cubic segment of the spline, between the data points $\mathbf{x}_i$ and $\mathbf{x}_{i+1}$, at $\mathbf{x}_i$ lies on Π_i and at $\mathbf{x}_{i+1}$ lies on the plane Π_{i+1}. Thus from the interpretations torsion preservation and coplanarity preservation criteria in sections 9 and 10 we observe that the Catmull-Rom splines (having tangent vectors with our magnitudes) preserve the torsion and coplanarity behavior of the data polygon.

14.1 Torsion and coplanarity preservation

Torsion preservation by cubic Catmull-Rom spline is also assured by the theorem 13.14 as follows. According to the theorem 13.14, the spline curve $\gamma(t)$ must satisfy the condition $[\mathbf{m}_{i-1}\ L_i\ \mathbf{m}_i]\triangle_i > 0$, where $\triangle_i = [L_{i-1}\ L_i\ L_{i+1}]$. But for Catmull-Rom spline we have $\mathbf{m}_i = \mathbf{x}_{i+1} - \mathbf{x}_{i-1} = L_i + L_{i+1}$, so that $[\mathbf{m}_{i-1}\ L_i\ \mathbf{m}_i] = [L_{i-1}\ L_i\ L_{i+1}] = \triangle_i$ and therefore the torsion preservation condition $[\mathbf{m}_{i-1}\ L_i\ \mathbf{m}_i]\triangle_i > 0$ holds $\forall i$. Also if $\mathbf{x}_{i-2}$, $\mathbf{x}_{i-1}$, $\mathbf{x}_i$, $\mathbf{x}_{i+1}$ are coplanar then $\tau_i(t) = [\mathbf{m}_{i-1}\ L_i\ \mathbf{m}_i] = [L_{i-1}\ L_i\ L_{i+1}] = 0$. Thus coplanarity condition is also satisfied.

14.2 Convexity and inflection preservation

It satisfies convexity and inflection preserving criteria with suitably chosen tangent length. This is mainly due to the reason that apart from the tangent vector at a data point $\mathbf{x}_i$ being coplanar with the plane Π_i the two consecutive sides L_i and L_{i+1} lies on one side of it.

14.3 Collinearity preservation

Also if the data points $\mathbf{x}_{i-1}$, $\mathbf{x}_i$ and $\mathbf{x}_{i+1}$ are collinear and $L_{i-1} \cdot L_i > 0$ then the tangent vector of the spline at $\mathbf{x}_i$ is collinear with $\mathbf{x}_{i-1}$, $\mathbf{x}_i$ and $\mathbf{x}_{i+1}$. Thus the Catmull-Rom splines having tangent vectors with our magnitudes satisfies collinearity preservation criteria.

We also observe that for collinear data arc, the shape of Catmull-Rom splines may not be aesthetically pleasing. We need to deviate from Catmull-Rom splines in accrodance with the conditions of modified definition of collinearity preservation criteria stated in section 8.

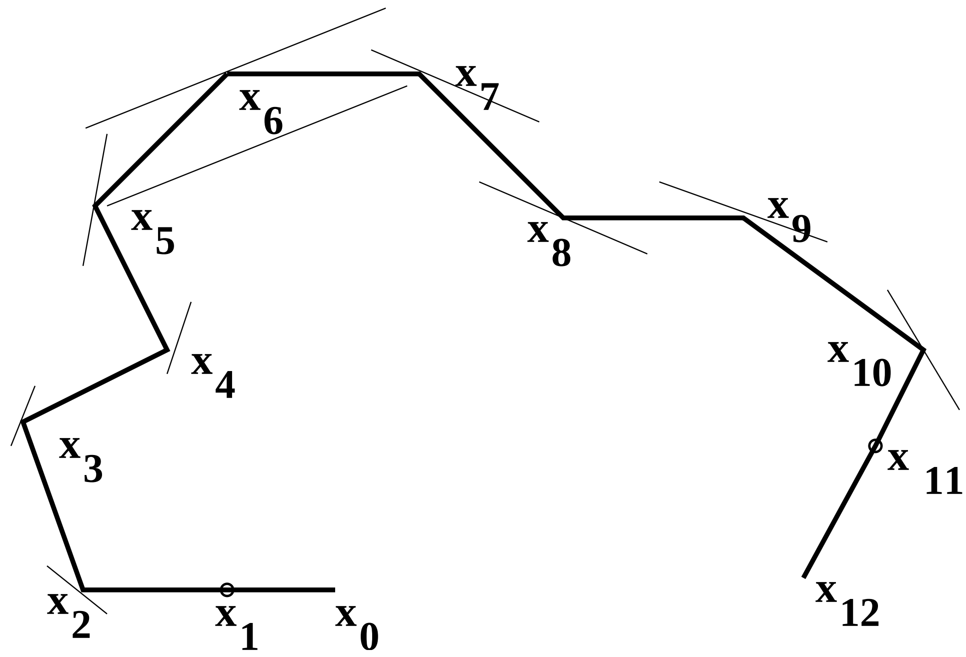


Figure 22: By suitable choice of magnitudes of tangent vectors of Catmull-Rom splines can preserve the shape of data polygon

15 Conclusion

We have analyzed the characterization for shape preservation criteria for splines. We have improved upon the definitive criteria for convexity preservation for splines. We have studied in detail the inflection criteria and various results concerning it. We have also stated the results from the literature which in conjunction with our analysis are observed to give negative results regarding convexity and inflection preservation criteria for splines. Such negative results would have been difficult to perceive intuitively. We have also discussed the analysis for collinearity, torsion and coplanarity preservation criteria.

We obtained a very important theorem 12.1, which states that there is a possibility that torsion preserving spline may need to have torsion set to zero at some nodepoints in order to be torsion continuous spline curve. From the literature we find that

- Shape preservation criteria also gives better way of segmentation of curves.
- Such curves can be very usefull tool for data reduction, which is very important for data transmission.
- Such interpolation can be used for robot path determination.

We are currently working towards getting profound theoretical and experimental results in the above directions.

We have found the characterization of all the shape preservation criteria for splines for the cubic case in terms of data points and slope vectors on them.

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